Constraint preserving boundary conditions in 3d linear gravity

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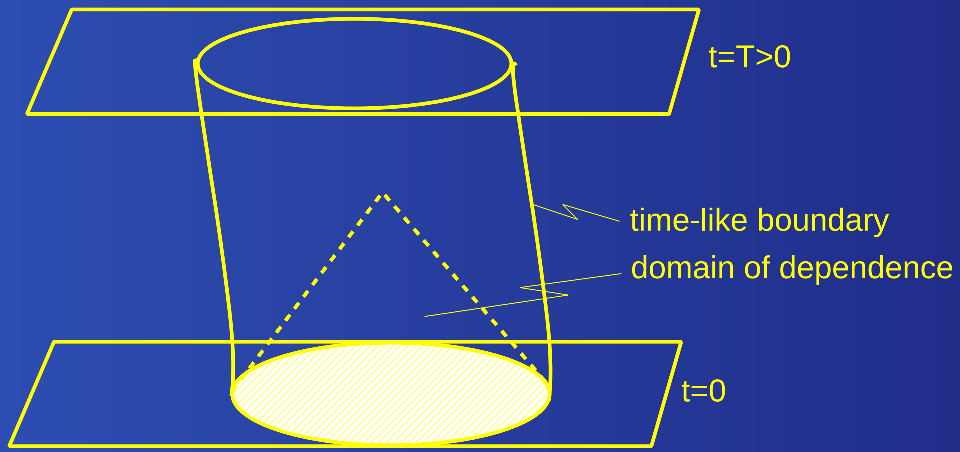
Introduction

Initial-boundary value problem in General Relativity:
Evolution of Einstein's equations in a "cylinder" $(t, \vec{x}) \in [0, T] \times \Omega$:



Initial data ($t = 0$) satisfies the constraints.
Bianchi identities only guarantee that the constraints are satisfied in the domain of dependence of the initial "region". Boundary data has to be given such that the constraints are satisfied in the whole cylinder!

Introduction

- Friedrich & Nagy: Constraint preserving bc for full nonlinear equations. Based on Weyl-like formulation with tetrads.
- Iriondo & Reula; Gioel, Luis & Manuel: Constraint preserving bc in spherically symmetry; with scalar field
- Schmidt, Szilagyi, Winicour; Szilagyi & Winicour: Similar techniques we use, well-posedness only for smooth boundaries.
- Stewart: Constraint preserving boundary conditions in the linearized regime.
- Bardeen & Buchmann

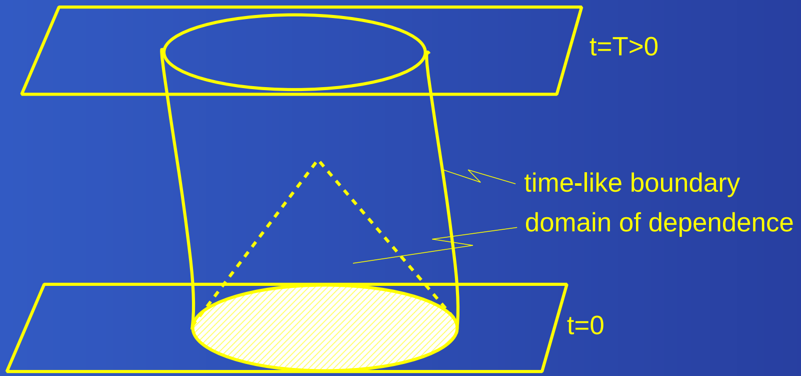
Today I am going to talk about how to implement constraint preserving bc in 3d linear gravity, with a cubic box as a domain Ω.

Symmetric hyperbolic systems

Consider a first order linear system:

$$\dot{u} = A^j \partial_j u + Bu, \qquad u = u(t, \vec{x}), t \geq 0, \vec{x} \in \Omega,$$

where the matrices A^1, A^2, A^3 are *symmetric*.



Initial data: $u(0, \vec{x}) = f(\vec{x})$.

Boundary data: $Mu(t, \vec{x}) = b(t, \vec{x})$, $\vec{x} \in \partial\Omega$.

Symmetric hyperbolic systems

Energy estimates:

$$E(t) = \int_{\Omega} (u, u) d^3x.$$

Initial-boundary value problem is called *well-posed* if

$$E(t) \leq C(T) \left(E(0) + \int_0^t \int_{\partial\Omega} (b, b) d\sigma d\tau \right), \qquad 0 \leq t \leq T,$$

where $C(T)$ does not depend on the initial data.

In particular, one has *uniqueness*.

Symmetric hyperbolic systems

Example: A^j symmetric and constant. $B = 0$:

$$\frac{d}{dt} E(t) = \int_\Omega 2(u, A^j \partial_j u)\, d^3x = \int_\Omega \partial_j (u, A^j u)\, d^3x = \int_{\partial\Omega} (u, A^i n_i u)\, d\sigma.$$

$A^n \equiv A^i n_i$: *boundary matrix*. Assume it has the eigenvalues $\pm 1, 0$. Decompose $u = u^{(+)} + u^{(-)} + u^{(0)}$ with $A^n u = u^{(+)} - u^{(-)}$:

$$(u, A^n u) = (u^{(+)}, u^{(+)}) - (u^{(-)}, u^{(-)}).$$

If we impose boundary conditions of the form $u^{(+)} = R u^{(-)}$ with R "small enough" so that $R^T R \leq 1$, we see that this terms is negative or zero and we obtain $E(t) \leq E(0)$.

Symmetric hyperbolic systems

Generalization to

$$u^{(+)} = Ru^{(-)} + b,$$

where b is a prescribed function at the boundary of the cylinder.
("Maximal dissipative boundary conditions").
If $b = 0$ we call the bc homogeneous.
If the boundaries are smooth, the existence of a smooth solution to the initial-boundary value problem for symmetric hyperbolic systems follows (Rauch, Secchi,...).

CPBC in the generalized EC system

Evolution equations: Generalized Einstein-Christoffel system, here in the weak field regime:

$$\begin{aligned} \partial_t K_{ij} &= \partial^k f_{kij} \,, \\ \partial_t f_{kij} &= \partial_k K_{ij} \,, \end{aligned}$$

f_{kij} is a linear combination of $\partial_k g_{ij}$.
Just like a wave equation for K_{ij} written in first order form...
Bunch of constraints. They depend linearly on spatial derivatives of the main variables K_{ij}, f_{kij}. Look at the evolution of the constraints.
Crucial point: Evolution system for the constraints should be symmetric (or symmetrizable) hyperbolic, so that we can apply the techniques described above.

CPBC in the generalized EC system

Assume domain has the shape of a cubic box.

Constraint preserving boundary conditions:

Step 1:

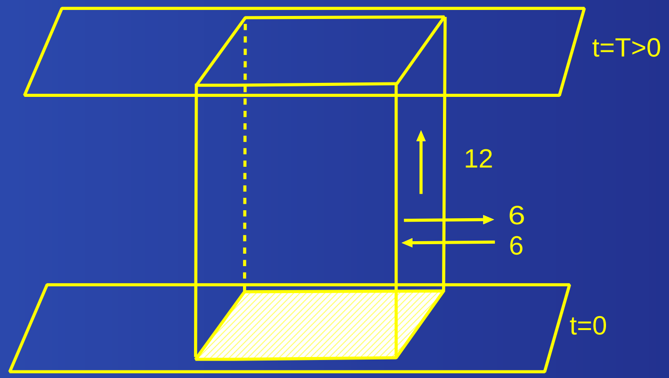
Characteristic variables (n^i: normal to the boundary, μ: eigenvalue):

$$\begin{aligned} \mu K &= n^k f_k \,, \\ \mu f_k &= n_k K \,, \end{aligned}$$

$$\begin{aligned} v^{(\pm)} &= K \pm f_n \,, \qquad (\mu = \pm 1), \\ v^{(0)}_k &= f_k - n_k f_n \,, \quad (\mu = 0). \end{aligned}$$

Classification: $v^{(\pm)} \to v^{(\pm)}_{phys}, v^{(\pm)}_{cv}, v^{(\pm)}_{gauge}$.

CPBC in the generalized EC system



Maximal dissipative boundary conditions:

$$v^{(+)} = T v^{(-)} + b, \qquad v^{(\pm)} = K \pm f_n\ .$$

Examples:

- $T = -1$: prescribing $K \sim \partial_t g$: Dirichlet conditions for g
- $T = 1$: prescribing $f_n \sim \partial_n g$: Neumann conditions for g

CPBC in the generalized EC system

Step 2:

Maximal dissipative bc for the evolution of the constraints:

If $V^{(\pm)}$ are the in- and outgoing characteristic variables for the evolution of the constraints, require

$$V^{(+)} = LV^{(-)},$$

where L is a coupling matrix, small enough for the energy estimate to hold.

Uniqueness implies that the constraints are satisfied everywhere provided they are satisfied initially.

CPBC in the generalized EC system

Step 3:
Express $V^{(\pm)}$ in terms of the main variables K, f_k:

$$V^{(\pm)} = \partial_n v_{cv}^{(\pm)} + \text{terms involving tangential derivatives of } K, f_k$$

Problem #1: We don't control normal derivatives at the boundary.
Solution: Use evolution equations

$$\partial_t v_{cv}^{(\pm)} = \pm \partial_n v_{cv}^{(\pm)} + \text{tangential derivatives of } K, f_k$$

and trade normal derivatives for time and tangential derivatives.
Problem #2: Are the resulting bc of maximal dissipative type?

$$0 = V^{(+)} - LV^{(-)} = \partial_t v_{cv}^{(+)} + L\partial_t v_{cv}^{(-)} + \text{tangential derivatives of } K, f_k$$

CPBC in the generalized EC system

Step 4:
Evolution of zero speed variables have the form

$$\partial_t v^{(0)} = \text{tangential derivatives of } K, f_k.$$

Find a set of linear combinations of in- and outgoing variables and zero speed variables $Z = (v_{cv}^{(+)} + \alpha\, v_{cv}^{(-)}, v^{(0)})$, and $F = (v_{phys}^{(+)} + \beta\, v_{phys}^{(-)}, v_{gauge}^{(+)} + \gamma\, v_{gauge}^{(-)})$ such that one has a closed evolution system at the boundary (∂_T: tangential derivatives):

(1) $$\partial_t Z = B^T \partial_T Z + C^T \partial_T F.$$

Here F is free data! The variables Z are determined by solving the boundary equation (1).

CPBC in the generalized EC system

This works only in two cases which correspond to the specification of Dirichlet or Neumann data to $g_{ij}^{(phys)}$.

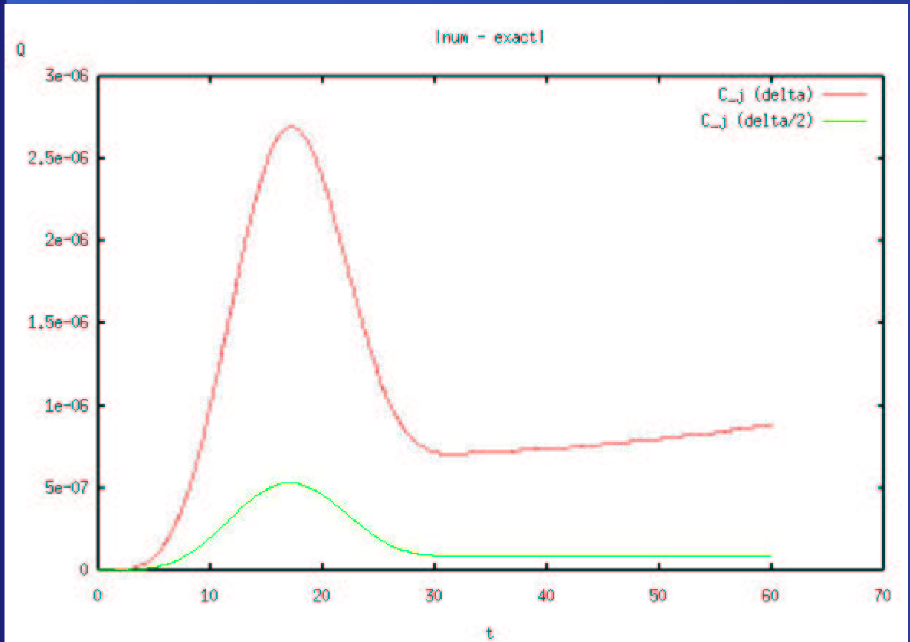
Summary:

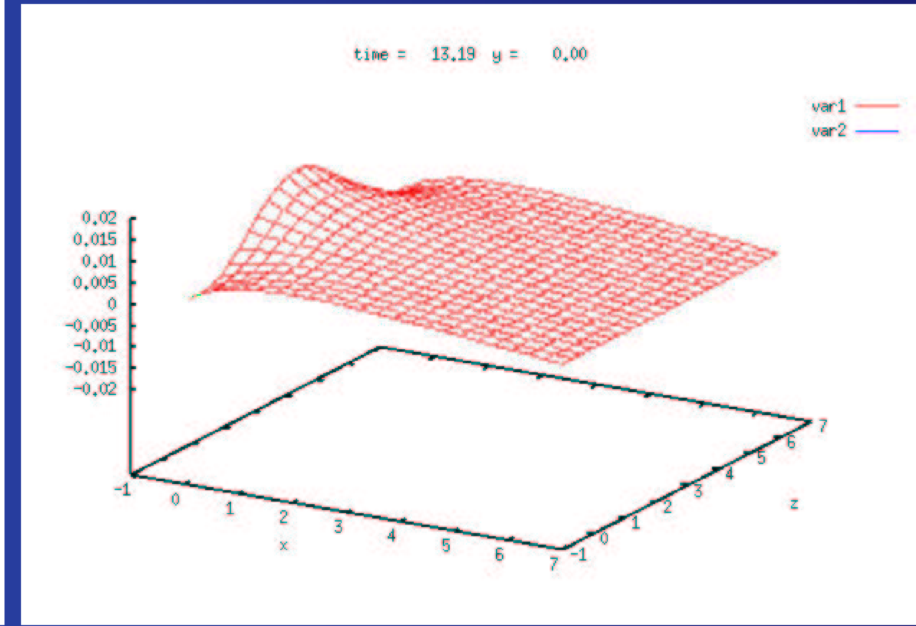
Well-posed constraint preserving bc can be given as follows:

- Provide data F ("physical and gauge functions") at the boundary. (Compatibility conditions at edges!)
- Obtain Z by evolving the boundary equations (which turns out to be symmetric hyperbolic).
- Since Z and F represent linear combinations of $v^{(+)}$ and $v^{(-)}$, they provide the data for the maximal dissipative bc for the main evolution system which may now be evolved. By construction, these bc guarantee that the constraints are satisfied beyond the domain of dependence of the initial region.

Numerical work...

...has started (Gioel, Manuel):





Discussion

- Derivation of well-posed constraint preserving boundary conditions for the generalized Einstein-Christoffel system in the weak field regime.
- Numerical implementation in progress (Gioel, Manuel)
- Generalization to more general backgrounds/nonlinear case?
- What is the physical meaning of our boundary conditions? Likely, “radiation” is going to be reflected...
- What should one give to the free source functions? Cauchy-characteristic or Cauchy perturbative methods...
- Higher order evolution scheme with the Weyl tensor as a fundamental variable might be more suitable for controlling radiation.