Particle infall using time domain Teukolsky formalism

Gaurav Khanna
Long Island University -Southampton

with
Ramon Lopez
U of Puerto Rico

INTRODUCTION

Particle in an equatorial, circular orbit about a Kerr black hole. Can also handle inclined, circular orbits.

Extensive past work on this: Detweiler, Poisson, Nakamura et al., Tanaka, Hughes, Finn and Thorne 2000.
However, all this past work has been based on frequency domain methods.

For GW Observatories, it is more useful to have time domain based information: waveforms, etc.

BH Perturbation Theory

Particle in orbit as a source of a perturbation of a Kerr black hole.

Teukolsky equation:

$$\begin{aligned}
&\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} \\
&+ \left[\frac{a^2}{\Delta} - \frac{1}{sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) \\
&-\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta}+\frac{i\,cos\theta}{sin^2\theta}\right]\frac{\partial\psi}{\partial\theta} \\
&-2s\left[\frac{M(a^2-r^2)}{\Delta} - r - i\,a\,cos\theta\right]\frac{\partial\psi}{\partial t} \\
&+\left[s^2cot^2\theta - s\right]\psi = 4\pi\Sigma T
\end{aligned}$$

Time Domain Method

Re-write the Teukolsky equation as a 2+1 PDE

by $\psi = \Psi(r,t,\theta)e^{im\phi}$

(Krivan, Laguna, Papadopoulos, Andersson, 1997)

$$-\frac{\partial^2\Phi}{\partial t^2} - A\frac{\partial\Phi}{\partial t} + b^2\frac{\partial^2\Phi}{\partial r^{*2}} + \frac{c}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Phi}{\partial\theta}\right) - V\Phi = T_f$$

where

$$\Phi = \Psi\sqrt{(r^2+a^2)\Delta^s}$$

$$A = \frac{1}{\Sigma^2}\left(2s\left[r\Delta - M(r^2+a^2)\right] + i\left[4Marm + 2sa\Delta cos\theta\right]\right)$$

$$b^2 = \frac{(r^2+a^2)^2}{\Sigma^2} \qquad c = \frac{\Delta}{\Sigma^2}$$

$$V = \frac{1}{\Sigma^2}(\Delta(s^2cot^2\theta - s) + m^2(\Delta sin^{-2}\theta - a^2) + 2sm\Delta cot\theta sin^{-1}\theta$$
$$-2ismar(r-M)) + f\frac{(r^2+a^2)}{\Sigma^2\Delta^s}\frac{\partial^2 f}{\partial r^{*2}}$$

$$f^2 = (r^2+a^2)\Delta^s$$

$$T_f = -\frac{4\pi\Sigma T f\Delta}{(r^2+a^2)^2 - a^2\Delta sin\theta}$$

SOURCE TERM

$$T = 2\rho^{-4} T_4$$

$$\begin{aligned} T_4 = {} & (\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)[(\delta^* - 2\tau^* + 2\alpha)T_{nm^*} \\ & - (\Delta + 2\gamma - 2\gamma^* + \mu^*)T_{m^*m^*}] + (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi) \\ & \times [(\Delta + 2\gamma + 2\mu^*)T_{nm^*} - (\delta^* - \tau^* + 2\beta^* + 2\alpha)T_{nn}] \end{aligned}$$

Delta, gamma, etc are various NP quantities with values as taken in a Kerr background, and ..

Tmm, Tm*m*, etc. are (Kinnersley) tetrad projections of the energy momentum tensor of a particle of mass mu and velocity u

$$T^{\mu\nu} = \frac{\mu}{\Sigma sin\theta dt/d\tau} u^{\mu} u^{\nu} \delta(r - r(t))\delta(\theta - \theta(t))\delta(\phi - \phi(t))$$

We next do a modal $e^{im\phi}$ decomposition of the azimuthal angular delta function, just as we did with the Teukolsky equation ...

And we approximate the other two delta functions as narrow gaussian distributions.

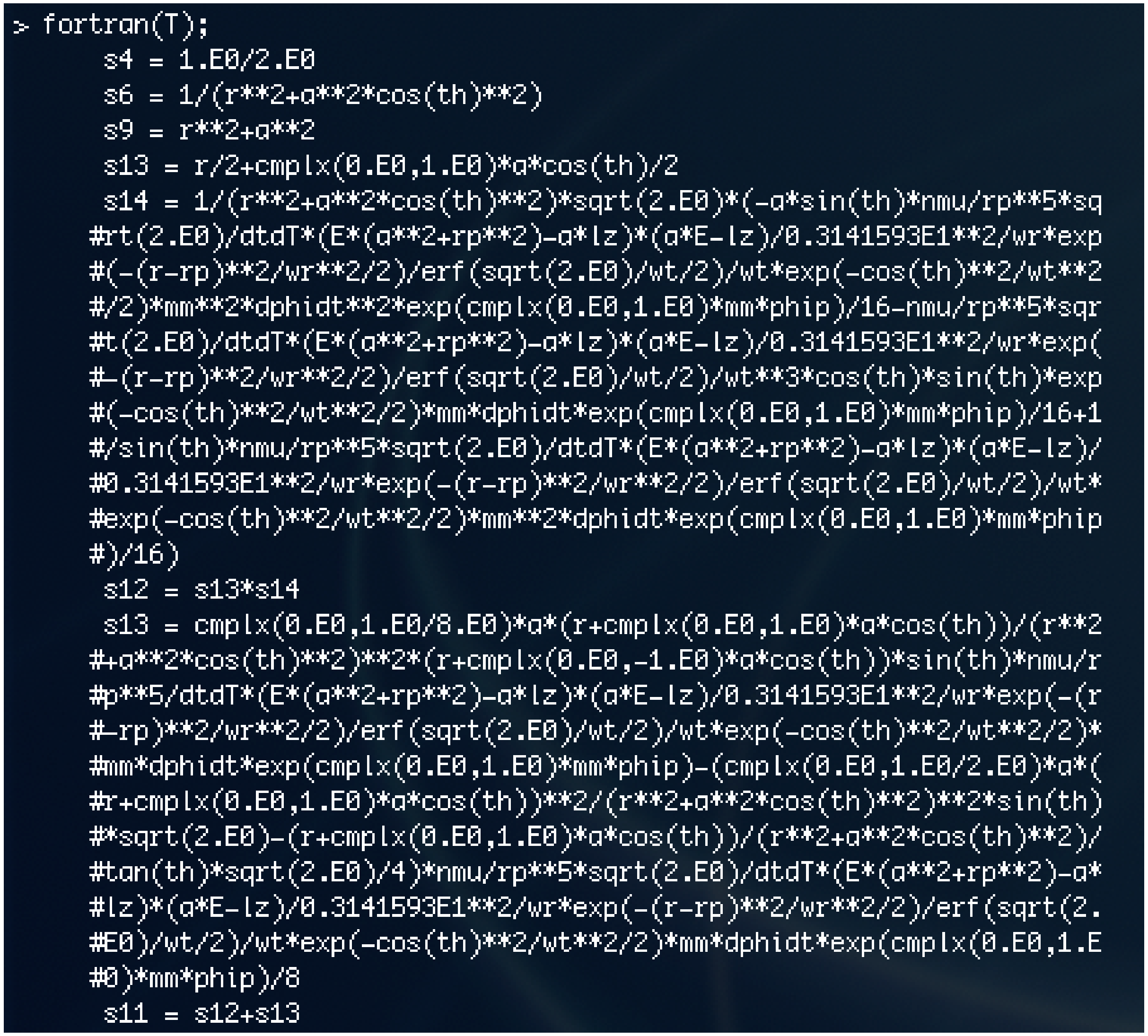
“Particle as a gaussian”

Radial and polar angular delta functions:

$$\delta(x - x(t)) \approx \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-(x - x(t))^2}{2\sigma^2} \right)$$

A better approach, as suggested by Lousto and Price shall be implemented in the future.

Glimpse of the source code



```
> fortran(T);
      s4 = 1.E0/2.E0
      s6 = 1/(r**2+a**2*cos(th)**2)
      s9 = r**2+a**2
      s13 = r/2+cmplx(0.E0,1.E0)*a*cos(th)/2
      s14 = 1/(r**2+a**2*cos(th)**2)*sqrt(2.E0)*(-a*sin(th)*nmu/rp**5*sq
     #rt(2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp
     #(-(r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2
     #/2)*mm**2*dphidt**2*exp(cmplx(0.E0,1.E0)*mm*phip)/16-nmu/rp**5*sqr
     #t(2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp(
     #-(r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt**3*cos(th)*sin(th)*exp
     #(-cos(th)**2/wt**2/2)*mm*dphidt*exp(cmplx(0.E0,1.E0)*mm*phip)/16+1
     #/sin(th)*nmu/rp**5*sqrt(2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-lz)/
     #0.3141593E1**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt*
     #exp(-cos(th)**2/wt**2/2)*mm**2*dphidt*exp(cmplx(0.E0,1.E0)*mm*phip
     #)/16)
      s12 = s13*s14
      s13 = cmplx(0.E0,1.E0/8.E0)*a*(r+cmplx(0.E0,1.E0)*a*cos(th))/(r**2
     #+a**2*cos(th)**2)**2*(r+cmplx(0.E0,-1.E0)*a*cos(th))*sin(th)*nmu/r
     #p**5/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp(-(r
     #-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)*
     #mm*dphidt*exp(cmplx(0.E0,1.E0)*mm*phip)-(cmplx(0.E0,1.E0/2.E0)*a*(
     #r+cmplx(0.E0,1.E0)*a*cos(th))**2/(r**2+a**2*cos(th)**2)**2*sin(th)
     #*sqrt(2.E0)-(r+cmplx(0.E0,1.E0)*a*cos(th))/(r**2+a**2*cos(th)**2)/
     #tan(th)*sqrt(2.E0)/4)*nmu/rp**5*sqrt(2.E0)/dtdT*(E*(a**2+rp**2)-a*
     #lz)*(a*E-lz)/0.3141593E1**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(sqrt(2.
     #E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)*mm*dphidt*exp(cmplx(0.E0,1.E
     #0)*mm*phip)/8
      s11 = s12+s13
```

```
      s12 = s11-1/(r**2+a**2*cos(th)**2)*(-(
     #*E-lz)**2/0.3141593E1**2/wr*exp(-(r-rp
     #/wt/2)/wt*exp(-cos(th)**2/wt**2/2)*mm*
     #1.E0)*mm*phip)/8+cmplx(0.E0,1.E0/8.E0)
     #/dtdT*(a*E-lz)**2/0.3141593E1**2/wr**3
     #/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(-cos(t
     #(cmplx(0.E0,1.E0)*mm*phip)+a*nmu/rp**4
     #1**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(sq
     #h)**2/wt**2/2)*mm**2*dphidt*exp(cmplx(
      s13 = s12+cmplx(0.E0,-1.E0/4.E0)*(-(r+
     #*2/(r**2+a**2*cos(th)**2)**3*(r+cmplx(
     #+a**2-2*M*r)/2+(r+cmplx(0.E0,1.E0)*a*c
     #2)**2*(r+cmplx(0.E0,-1.E0)*a*cos(th))*
     #-lz)**2/0.3141593E1**2/wr*exp(-(r-rp)*
     #t/2)/wt*exp(-cos(th)**2/wt**2/2)*mm*dp
     #*phip)
      s10 = s13+cmplx(0.E0,1.E0/4.E0)*(-(r+c
     #*2/(r**2+a**2*cos(th)**2)**3*(r+cmplx(
     #a**2-2*M*r)/2+(r+cmplx(0.E0,1.E0)*a*co
     #)**2*(r+cmplx(0.E0,-1.E0)*a*cos(th))*(
     #lz)**2/0.3141593E1**2/wr*exp(-(r-rp)**
     #/2)/wt*exp(-cos(th)**2/wt**2/2)*mm*dph
     #phip)+cmplx(0.E0,1.E0/16.E0)*(r+cmplx(
     #**2+a**2*cos(th)**2)**3*(r+cmplx(0.E0,
     #2*M*r)*nmu/rp**4/dtdT*(a*E-lz)**2/0.31
     #/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(
     #dt*exp(cmplx(0.E0,1.E0)*mm*phip)
      s8 = s9*s10
      s11 = -1
      s13 = r**2+a**2-2*M*r
      s19 = 1/(r**2+a**2*cos(th)**2)/2
      s20 = sqrt(2.E0)*(cmplx(0.E0,1.E0/16.E
     #2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-
     #r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)
     #*mm*dphidt*exp(cmplx(0.E0,1.E0)*mm*phi
     #u/rp**5*sqrt(2.E0)/dtdT*(E*(a**2+rp**2
     #**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(sqr
     #sin(th)*exp(-cos(th)**2/wt**2/2)*exp(cmplx(0.E0,1.E0)*mm*phip)+
     #lx(0.E0,-1.E0/16.E0)/sin(th)*nmu/rp**5*sqrt(2.E0)/dtdT*(E*(a**2
     #**2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp(-(r-rp)**2/wr**2/2)/e
     #sqrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)*mm*exp(cmplx(0.E0,
     #0)*mm*phip))
      s18 = s19*s20
```

```
      s20 = -(r+cmplx(0.E0,1.E0)*a*cos(th))/(r**2+a**2*cos(th)**2)**2
      s21 = sqrt(2.E0)*(cmplx(0.E0,1.E0/16.E0)*a*sin(th)*nmu/rp**5*sqrt(
     #2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp(-(
     #r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)
     #*mm*dphidt*exp(cmplx(0.E0,1.E0)*mm*phip)+cmplx(0.E0,1.E0/16.E0)*nm
     #u/rp**5*sqrt(2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-lz)/0.3141593E1
     #**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt**3*cos(th)*
     #sin(th)*exp(-cos(th)**2/wt**2/2)*exp(cmplx(0.E0,1.E0)*mm*phip)+cmp
     #lx(0.E0,-1.E0/16.E0)/sin(th)*nmu/rp**5*sqrt(2.E0)/dtdT*(E*(a**2+rp
     #**2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(
     #sqrt(2.E0)/wt/2)/wt*
     #0)*mm*phip))*r
      s19 = s20*s21
      s17 = s18+s19
      s19 = s17
      s21 = r/2+cmplx(0.E
      s23 = 1/(r**2+a**2*
      s25 = sqrt(2.E0)
      s26 = cmplx(0.E0,-1.
     #(E*(a**2+rp**2)-a*lz
     #-rp)**2/wr**2/2)/erf
     #mm*dphidt*exp(cmplx(
     #u/rp**5*sqrt(2.E0)/
     #**2/wr**3*(r-rp)*exp
     #3*cos(th)*sin(th)*ex
     #*phip)+cmplx(0.E0,1.
     #*(a**2+rp**2)-a*lz)*
     #p)**2/wr**2/2)/erf(
     #*exp(cmplx(0.E0,1.E
      s24 = s25*s26
      s22 = s23*s24
      s20 = s21*s22
      s18 = s19+s20
      s16 = s18+a/(r**2+a*
     #h))*sin(th)*nmu/rp**
     #3E1**2/wr*exp(-(r-rp
     #(th)**2/wt**2/2)*exp
     #.E0)*a*cos(th))/(r**
     #os(th))*sin(th)*nmu/
     #41593E1**2/wr*exp(-(
     #-cos(th)**2/wt**2/2)
```

```
      s15 = sqrt(2.E0)*(cmplx(0.E0,1.E0/16.E0)*a*sin(th)*nmu/rp**5*sqrt(
     #2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp(-(
     #r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)
     #*mm*dphidt*exp(cmplx(0.E0,1.E0)*mm*phip)+cmplx(0.E0,1.E0/16.E0)*nm
     #u/rp**5*sqrt(2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E-lz)/0.3141593E1
     #**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt**3*cos(th)*
     #sin(th)*exp(-cos(th)**2/wt**2/2)*exp(cmplx(0.E0,1.E0)*mm*phip)+cmp
     #lx(0.E0,-1.E0/16.E0)/sin(th)*nmu/rp**5*sqrt(2.E0)/dtdT*(E*(a**2+rp
     #**2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(
     #sqrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)*mm*exp(cmplx(0.E0,1.E
     #0)*mm*phip))
      s13 = s14*s15
      s11 = s12*s13
      s12 = a*(r+cmplx(0.E0,1.E0)*a*cos(th))/(r**2+a**2*cos(th)**2)**2*(
     #r+cmplx(0.E0,-1.E0)*a*cos(th))*sin(th)*nmu/rp**5/dtdT*(E*(a**2+rp*
     #*2)-a*lz)*(a*E-lz)/0.3141593E1**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(s
     #qrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)*exp(cmplx(0.E0,1.E0)*m
     #m*phip)/8+cmplx(0.E0,1.E0/8.E0)*(cmplx(0.E0,1.E0/2.E0)*a*(r+cmplx(
     #0.E0,1.E0)*a*cos(th))**2/(r**2+a**2*cos(th)**2)**2*sin(th)*sqrt(2.
     #E0)-(r+cmplx(0.E0,1.E0)*a*cos(th))/(r**2+a**2*cos(th)**2)/tan(th)*
     #sqrt(2.E0)/4)*nmu/rp**5*sqrt(2.E0)/dtdT*(E*(a**2+rp**2)-a*lz)*(a*E
     #-lz)/0.3141593E1**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2
     #)/wt*exp(-cos(th)**2/wt**2/2)*exp(cmplx(0.E0,1.E0)*mm*phip)
      s10 = s11+s12
      s11 = s10-1/(r**2+a**2*cos(th)**2)*(cmplx(0.E0,1.E0/8.E0)*(r**2+a*
     #*2)*nmu/rp**4/dtdT*(a*E-lz)**2/0.3141593E1**2/wr*exp(-(r-rp)**2/wr
     #**2/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)*mm*dphidt*
     #exp(cmplx(0.E0,1.E0)*mm*phip)+(r**2+a**2-2*M*r)*nmu/rp**4/dtdT*(a*
     #E-lz)**2/0.3141593E1**2/wr**3*(r-rp)*exp(-(r-rp)**2/wr**2/2)/erf(s
     #qrt(2.E0)/wt/2)/wt*exp(-cos(th)**2/wt**2/2)*exp(cmplx(0.E0,1.E0)*m
     #m*phip)/8+cmplx(0.E0,-1.E0/8.E0)*a*nmu/rp**4/dtdT*(a*E-lz)**2/0.31
     #41593E1**2/wr*exp(-(r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(
     #-cos(th)**2/wt**2/2)*mm*exp(cmplx(0.E0,1.E0)*mm*phip))/2
      s12 = s11-(-(r+cmplx(0.E0,1.E0)*a*cos(th))**2/(r**2+a**2*cos(th)**
     #2)**3*(r+cmplx(0.E0,-1.E0)*a*cos(th))*(r**2+a**2-2*M*r)/2+(r+cmplx
     #(0.E0,1.E0)*a*cos(th))/(r**2+a**2*cos(th)**2)**2*(r+cmplx(0.E0,-1.
     #E0)*a*cos(th))*(r-M)/2)*nmu/rp**4/dtdT*(a*E-lz)**2/0.3141593E1**2/
     #wr*exp(-(r-rp)**2/wr**2/2)/erf(sqrt(2.E0)/wt/2)/wt*exp(-cos(th)**2
     #/wt**2/2)*exp(cmplx(0.E0,1.E0)*mm*phip)/4
```

Back Reaction

Adiabatic orbit decay aka “Poor man’s approach”

Circular orbits are parametrized by **E, Lz, Q.**
Since circular orbits decay to circular orbits, the path of a particle inspiraling into a BH is a path in (E,Lz,Q) “phase” space.

Our Method: Start with a circular orbit (E,Lz,Q); we calculate the energy and ang. mom. loss via gravitational waves for a few full orbits; we correct the orbit using the new values of (E,Lz,Q).

Calculate energy and ang. mom. loss using ..

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{r\to\infty}\left\{\frac{1}{4\pi r^6}\int_\Omega \mathrm{d}\Omega \left|\int_{-\infty}^{t} \mathrm{d}\tilde{t}\,\psi_4(\tilde{t},r,\theta,\varphi)\right|^2\right\}$$

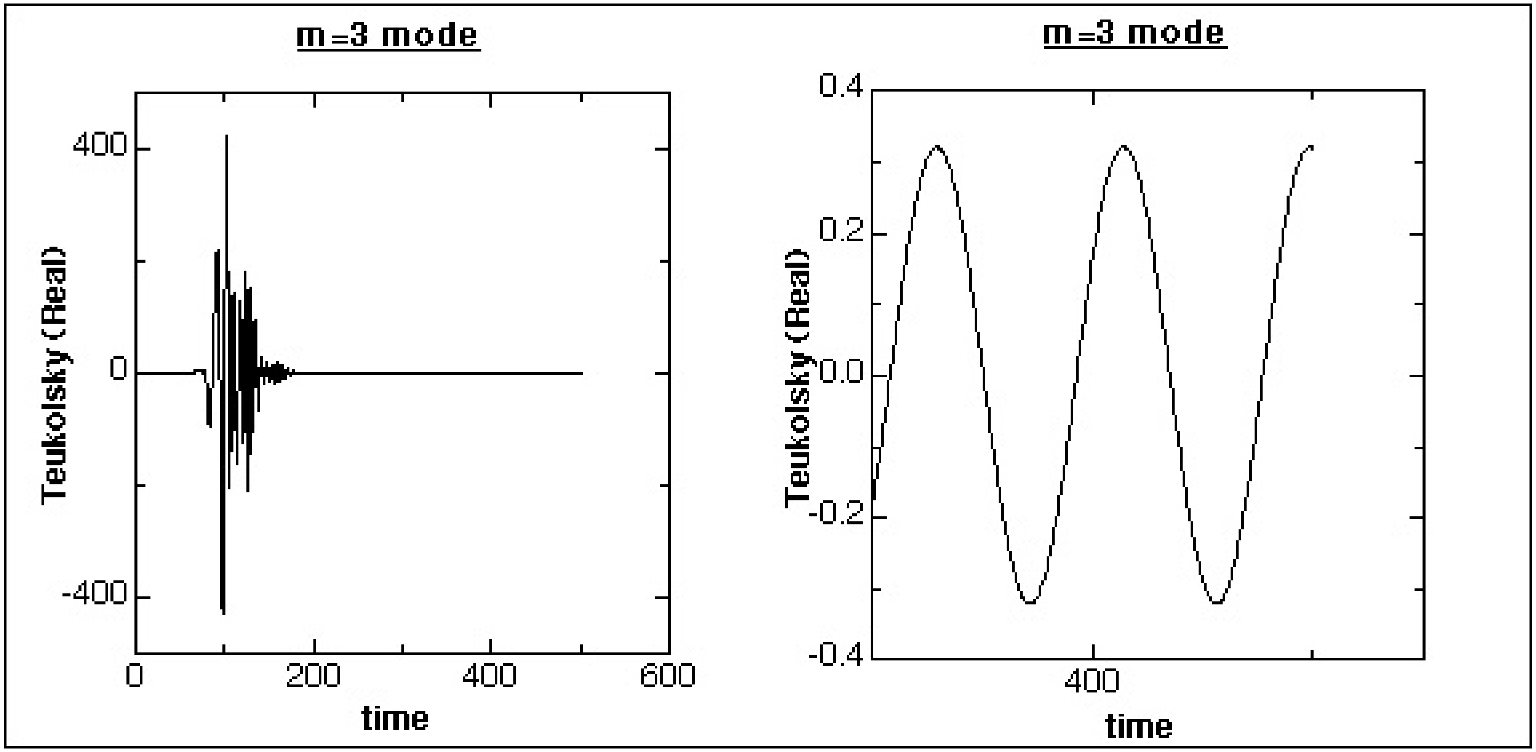
$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = -\lim_{r\to\infty}\left\{\frac{1}{4\pi r^6}\mathbf{Re}\left[\int_\Omega \mathrm{d}\Omega\left(\partial_\varphi \int_{-\infty}^{t} \mathrm{d}\tilde{t}\,\psi_4(\tilde{t},r,\theta,\varphi)\right)\right.\right.$$

$$\left.\left.\times\left(\int_{-\infty}^{t}\mathrm{d}t'\int_{-\infty}^{t'}\mathrm{d}\tilde{t}\,\bar{\psi}_4(\tilde{t},r,\theta,\varphi)\right)\right]\right\} \qquad \mathrm{d}\Omega = \sin\theta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$$

Update, E, Lz and also Q (for circular orbits, change in E and Lz determines change in Q)

RESULTS

Waveforms for m=3 mode (equatorial circular orbit radius r/M=11.6 about a Kerr hole with a/M = 0.9)



Energy Fluxes

and comparison with frequency domain results (Finn & Thorne 2000)

mode	1	2	3	4
TCS	2.1×10^{-12}	1.8×10^{-09}	2.0×10^{-10}	2.0×10^{-11}
FT	2.4×10^{-12}	2.2×10^{-09}	2.2×10^{-10}	2.7×10^{-11}

EXCELLENT AGREEMENT!

FUTURE WORK

Implementation of back reaction.

Run the code to death .. with different parameters, orbits, etc.

Implement inclined circular orbits .. run that code to death.

Implement other better approaches to model the particle .. a la Lousto Price .. and expect even better agreement with frequency domain results!