

A Radiation Scalar for Numerical Relativity

Chris Beetle

University of Utah

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Work with Lior Burko.

Summary

Idea: Extract partial information about gravitational radiation using a scalar curvature invariant.

- Where “gravitational radiation” has unambiguous meaning, define

$$\xi := \Psi_0 \Psi_4 \quad \text{with} \quad \Psi_1 = 0 = \Psi_3.$$

- The definition acquires meaning in a *transverse frame*.
- This choice (almost) always determines an (almost) unique frame.
- While the result does have limitations, it has advantages as well

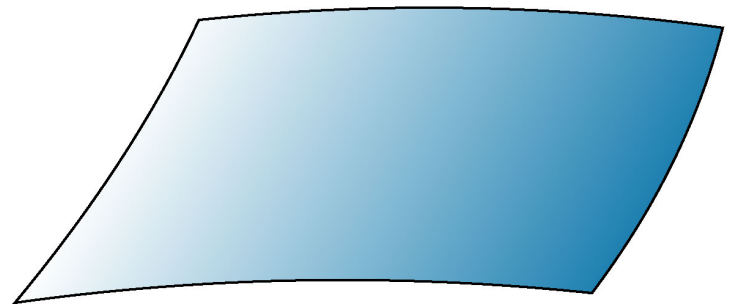
- | | |
|---|---|
| | + generically applicable and sensitive |
| – not universally applicable | + does not require perturbation theory |
| – insensitive to certain radiation fields | + clear interpretation in well-understood radiation zones |
| | + smoothly extends to all space-time. |

Gravitational Radiation

Gravitational radiation is not defined everywhere in space-time;
geometry must vary on *two* distinct length scales.

Example: Linearized Radiation Fields

- Space-time is nearly Minkowski.
(infinite length scale)

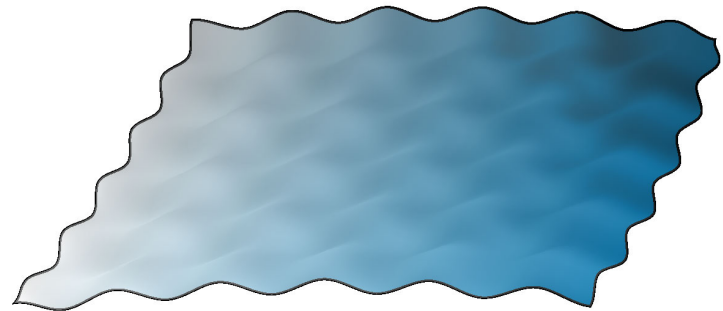


Gravitational Radiation

Gravitational radiation is not defined everywhere in space-time; geometry must vary on *two* distinct length scales.

Example: Linearized Radiation Fields

- Space-time is nearly Minkowski. (infinite length scale)
- Weak field perturbations propagate. (wavelength)



As in Maxwell theory, regions where gravitational radiation can be unambiguously defined are known as *radiation* (or *wave*) *zones*.

Unlike Maxwell theory, gravitational radiation in such zones must generally be *weak*.

Radiation Zones

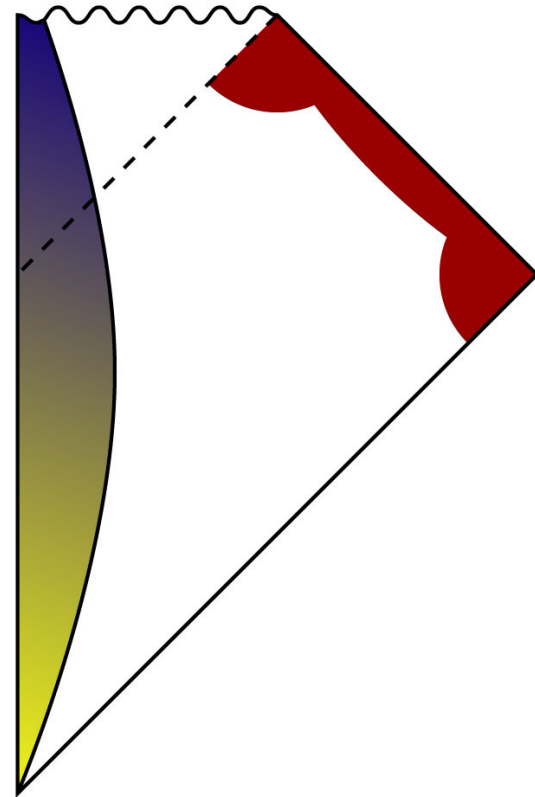
The gravitational field in a radiation zone is dominated by a Coulombic part.

Example: A star collapses to form a black hole, then comes to equilibrium.

- The large-distance limit is nearly Minkowski.
- The late-time limit is nearly Schwarzschild (or Kerr).

Weak radiation in both regions can be described using perturbation theory.

$$g_{ab} = \overset{\circ}{g}_{ab} + \delta g_{ab}$$



Issues for Numerical Relativity

Q How does one recognize a wave zone using only the physical metric?

- Equilibrium black hole space-times are algebraically special.

$$\mathcal{S} := \frac{27 J^2}{I^3} = 1 \quad \text{with} \quad \begin{aligned} I &:= \frac{1}{16} (C_{ab}{}^{cd} C_{cd}{}^{ab} - i C_{ab}{}^{cd} \star C_{cd}{}^{ab}) \\ J &:= \frac{1}{96} (C_{ab}{}^{cd} C_{cd}{}^{ef} C_{ef}{}^{ab} - i C_{ab}{}^{cd} C_{cd}{}^{ef} \star C_{ef}{}^{ab}) \end{aligned}$$

⇒ The *speciality index* \mathcal{S} approaches one in the late-time wave zone.
(Baker and Campanelli, 2000)

! I and J approach zero in the large-distance limit; $\mathcal{S} \sim \text{const.} \neq 1$.

Q How can one split the physical metric into background and perturbation?

! Even if space-time is nearly algebraically special in a *finite* region, there are many possible background geometries.

Radiation Scalars

- An alternative approach is to seek quantities which
 1. can be defined directly in terms of the physical geometry,
 2. are independent of any coordinate system, and
 3. depend only on the radiation field in the wave zone.

+ The first two properties are satisfied by any Weyl curvature invariant.

$$I := \frac{1}{16} (C_{ab}{}^{cd} C_{cd}{}^{ab} - i C_{ab}{}^{cd} \star C_{cd}{}^{ab})$$

$$J := \frac{1}{96} (C_{ab}{}^{cd} C_{cd}{}^{ef} C_{ef}{}^{ab} - i C_{ab}{}^{cd} C_{cd}{}^{ef} \star C_{ef}{}^{ab}).$$

- While I and J are the simplest Weyl invariants, neither vanishes in an equilibrium black hole space-time.
- We call a curvature invariant with all three properties a *radiation scalar*.

Newman–Penrose Formalism

- Choose a null frame $(\ell^a, n^a, m^a, \bar{m}^a)$ with $\ell^a n_a = -1 = -m^a \bar{m}_a$.
- The Newman–Penrose components Ψ_n of the Weyl tensor are

$\Psi_0 := C_{abcd} \ell^a m^b \ell^c m^d$	\leftrightarrow	Transverse radiation along n^a
$\Psi_1 := C_{abcd} \ell^a m^b \ell^c n^d$	\leftrightarrow	Longitudinal radiation along n^a
$\Psi_2 := C_{abcd} \ell^a m^b \bar{m}^c n^d$	\leftrightarrow	Coulombic field
$\Psi_3 := C_{abcd} \ell^a n^b \bar{m}^c n^d$	\leftrightarrow	Longitudinal radiation along ℓ^a
$\Psi_4 := C_{abcd} \bar{m}^a n^b \bar{m}^c n^d$	\leftrightarrow	Transverse radiation along ℓ^a
- In the wave zones, the transverse components describe physical radiation; the longitudinal components are gauge.

One can *shift* gauge from space-time coordinates to the frame.

Transverse Frames

Since the longitudinal components are pure gauge in a wave zone, focus on *transverse frames* in which those components vanish.

$$\psi_1 = 0 = \psi_3$$

This is natural both near null infinity and at late times (Teukolsky).

- Such frames always exist in physically relevant situations. (Type I, Type D, Type N)

- In those situations, only finitely many are interesting.

Type I Three distinct frames.

Type D Infinitely many frames, but only one Kinnersley frame.

Type II One frame.

The Transverse Radiation Scalar

- In each of the (typically) three distinct transverse frames, define

$$\xi := \Psi_0 \Psi_4.$$

+ In the Kinnersley frame of a type II or type D geometry, ξ vanishes.

⇒ This value, ξ^0 , is a radiation scalar.

Q How can one evaluate ξ^0 ?

- The (generically) three values of ξ are the roots of the polynomial

$$P := 16\xi^3 - 24I\xi^2 + 9I^2\xi + 27J^2 - I^3 = 0.$$

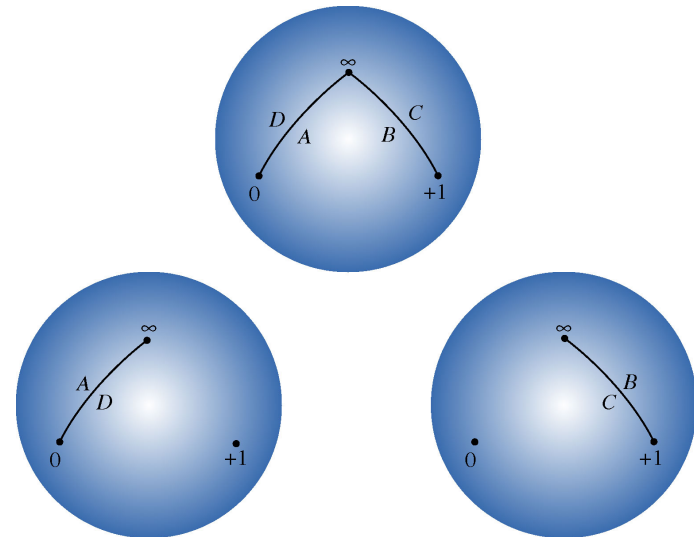
Calculating the Radiation Scalar

The Cardano formula gives the roots of P :

$$\xi = I Z(S) := \frac{I}{4} \left(2 - \sqrt[3]{2S - 1 + 2\sqrt{S^2 - S}} - \sqrt[3]{2S - 1 - 2\sqrt{S^2 - S}} \right).$$

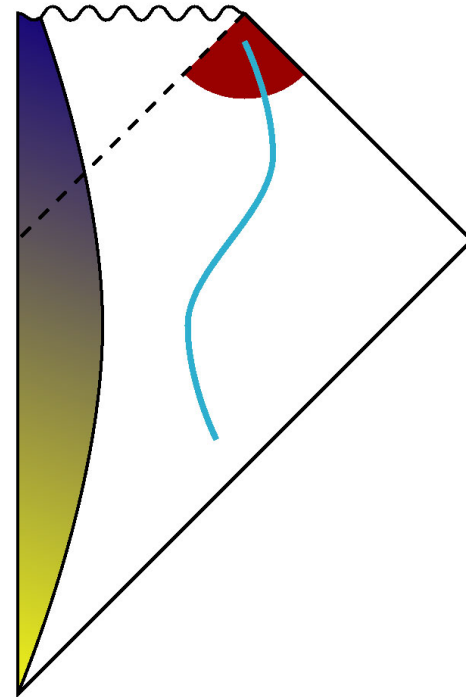
The three branches of the cube root yield the three values of ξ .

- The Riemann surface for $Z(S)$ has an interesting structure.
- + The leaf where $Z(S)$ is regular at $S = 1$ gives ξ^0 in the wave zone.
- The branch structure of $Z(S)$ implies ξ^0 is discontinuous.



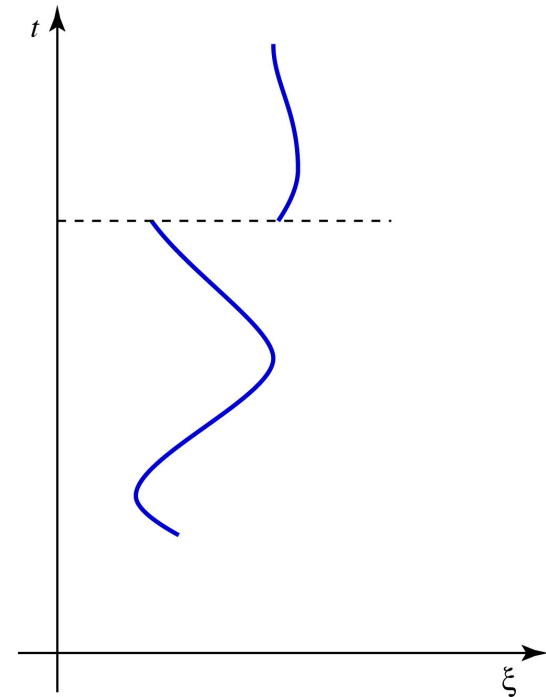
Extension to Strong Field Regions

1. Pick a curve extending to late times.



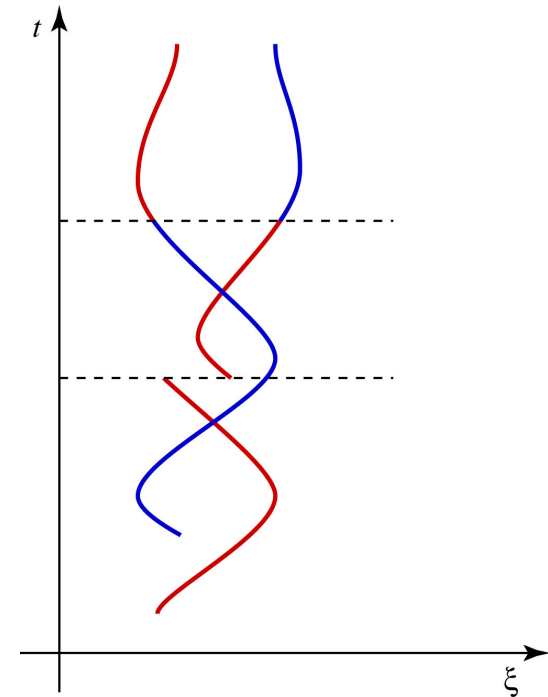
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3. ξ^0 may be discontinuous at points.



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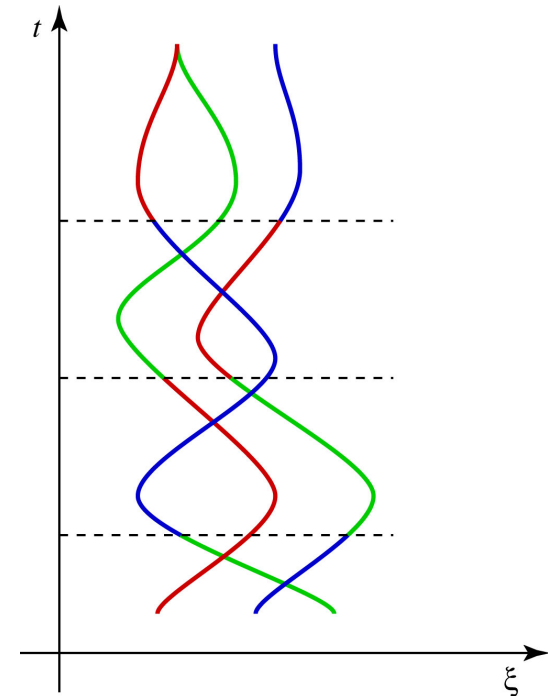
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4. At such points, it is continuous with ξ^\pm .
5. Define $\xi(t) := \xi^\pm$ at earlier times.
6. Repeat this process, keeping $\xi(t)$ smooth.

⇒ Track all roots, and decide afterward which is relevant at a given point.



The function $\xi(t)$ gives a *smooth* extension of ξ^0 to all of space-time.

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