A new symmetric hyperbolic formulation for the Einstein equations

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The Einstein equations as PDEs in 3+1 variables

$$\partial_t h_{ij} = -2ak_{ij} + 2D_{(i}b_{j)}, \qquad \text{(E1)}$$

$$\partial_t k_{ij} = a[R_{ij} + (k_l^l)k_{ij} - 2k_{il}k_j^l] + b^l D_l k_{ij} \qquad \text{(E2)}$$
$$+ k_{il}D_j b^l + k_{lj}D_i b^l - D_i D_j a,$$

$$R_i^i + (k_i^i)^2 - k_{ij}k^{ij} = 0, \qquad \text{(HC)}$$

$$D^j k_{ij} - D_i k_j^j = 0. \qquad \text{(MC)}$$

spatial metric $h \in \mathbb{S}$, extrinsic curvature $k \in \mathbb{S}$, lapse $a \in \mathbb{R}$, shift $b \in \mathbb{R}^3$, all depend on $x \in \mathbb{R}^3$, $t \geq 0$.

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Initial value problem: Given lapse and shift and given h and k for $t = 0$ satisfying (HC) and (MC), solve the evolution equations (E1) and (E2) to find h and k for $t \geq 0$.

Linearize about flat space in Cartesian coordinates:

$$h_{ij} = \delta_{ij} + \gamma_{ij}, \qquad k_{ij} = 0 + \kappa_{ij}$$
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$$\partial_t \gamma_{ij} = -2\kappa_{ij} + 2\partial_{(i}\beta_{j)}, \tag{e1}$$

$$\partial_t \kappa_{ij} = (P\gamma)_{ij} - \partial_i \partial_j \alpha, \tag{e2}$$

$$(P\gamma)_i^i = 0, \tag{hc}$$

$$(M\kappa)_i = 0. \tag{mc}$$

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$$(P\gamma)_{ij} := \frac{1}{2}\partial_i \partial^l \gamma_{lj} + \frac{1}{2}\partial_j \partial^l \gamma_{li} - \frac{1}{2}\partial^l \partial_l \gamma_{ij} - \frac{1}{2}\partial_i \partial_j \gamma_l^l$$

$$(M\kappa)_i := \partial^j \kappa_{ij} - \partial_i \kappa_j^j$$

N.B.: $(P\gamma)_i^i = \partial^i (M\gamma)_i$

A key identity

For any symm. γ,

$$(P\gamma)_{ij} = -\partial^l[\partial_{[l}\gamma_{j]i} + (M\gamma)_{[l}\delta_{j]i}] + \frac{1}{2}\partial^l(M\gamma)_l\delta_{ij}.$$

This identity relates the linearized Ricci tensor, the linearized momentum constraint operator and the linearized Hamiltonian constraint operator.

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Define $L : C^\infty(\mathbb{R}^3, \mathbb{S}) \to C^\infty(\mathbb{R}^3, \mathbb{T})$ by $(Lu)_{lji} = \partial_{[l}u_{j]i}$ w/
$\mathbb{T} = \{\, w \in \mathbb{R}^{3\times3\times3} \,|\, w_{ijk} + w_{jik} = 0, w_{ijk} + w_{jki} + w_{kij} = 0 \,\}$
$\dim \mathbb{T} = 8$

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or

$$(P\gamma)_{ij} = \sqrt{2}\partial^l\lambda_{l(ij)} + \frac{1}{2}\partial^l(M\gamma)_l\delta_{ij}$$

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$$(P\gamma)_{ij} = \sqrt{2}\partial^l \lambda_{l(ij)} + \frac{1}{2}\partial^l (M\gamma)_l \delta_{ij} = \sqrt{2}\partial^l \lambda_{l(ij)}$$

(hc)

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Thus

$$\partial_t\kappa_{ij} = (P\gamma)_{ij} - \partial_i\partial_j\alpha \qquad \text{(e2)}$$

becomes

$$\partial_t\kappa_{ij} = -\sqrt{2}(L^*\lambda)_{ij} - \partial_i\partial_j\alpha$$

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This gives

$$\partial_t \lambda_{lji} = \sqrt{2}(L\kappa)_{lji} - \tau_{lji}$$

where

$$\tau_{lji} = \frac{1}{\sqrt{2}}(\partial_i \partial_{[l}\beta_{j]} + \partial^m \partial_{[m}\beta_{l]}\delta_{ij} - \partial^m \partial_{[m}\beta_{j]}\delta_{il})$$

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$$\partial_t \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 & L \\ -L^* & 0 \end{pmatrix} \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} - \begin{pmatrix} \partial\partial\alpha \\ \tau \end{pmatrix}$$

FOSH

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Given initial data for γ and κ, initialize λ in the obvious way:

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Once κ is known, recover γ by integration:

$$\gamma_{ij} = \gamma_{ij}(0) - 2\int_0^t (\kappa_{ij} - \partial_{(i}\beta_{j)})$$

Theorem: *Let* α *and* β *be given and suppose that* $\gamma(0)$ *and* $\kappa(0)$ *are given satisfying* (hc) *and* (mc). *Define* $\lambda(0)$ *as above. Determine* λ *and* κ *by the FOSH system. Determine* γ *by integration. Then* (e1), (e2), (hc), (mc) *are satisfied.*

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Trickiest part is to show that $\mu := M\kappa \equiv 0$. In fact

$$\partial_t^2 \mu_i = \partial^l \partial_{(i} \mu_{l)}, \qquad \mu(0) = \partial_t \mu(0).$$

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This elastic wave equation is well-posed so the solution is trivial.

In this case we define metric-dependent linear operators $L : C^\infty(\mathbb{R}^3, \mathbb{S}) \to C^\infty(\mathbb{R}^3, \mathbb{T})$, $L^* : C^\infty(\mathbb{R}^3, \mathbb{T}) \to C^\infty(\mathbb{R}^3, \mathbb{S})$

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We again introduce 8 new variables:

$$f_{lmn} = -\frac{1}{\sqrt{2}}[(Lh)_{lmn} + (Mh)_{[l}h_{m]n}]$$

Find $k(x,t) \in \mathbb{S}$, $f(x,t) \in \mathbb{T}$ such that

$$\partial_0 k_{ij} = -\sqrt{2} a (L^* f)_{ij} + B_{ij}$$
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$\partial_0 := \partial_t - b^l \partial_l$

B_{ij} and C_{lmn} are algebraic combinations of h_{ij}, $\partial_l h_{ij}$, k_{ij}, the lapse a and the shift b and their spatial derivatives.

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The system is coupled to

$$\partial_0 h_{ij} = -2ak_{ij} + 2h_{s(i}\partial_{j)}b^s$$

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- Fritelli–Reula, Einstein–Christoffel, and Kidder–Scheel-Teukolsky introduce $d_{ijk} = \partial_i h_{jk} + \ldots$ as 18 additional unknowns. We introduce only 8.

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- In an earlier version I spoke about we introduced only 6 additional unknowns. However in that version $\partial_t \kappa$ is a fundamental unknown rather than κ, and γ has to be recovered by two integrations. This seems more "natural".
- We can identify elements of $\mathbb{T}$ with trace-free matrices. Then L and L^* become variants of curl operators.

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Future work. . .

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- Computations. . . Does this help get a more stable evolution?