Gauge fixing:
Elliptic gauge conditions and constrained evolution

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Introduction

We want to collide black holes. This is difficult. In 2000, many codes were unstable.

In 2000, Penn State thus created a "back to the basics" concept: don't try to fiddle with the equations; instead, do simple things, and understand them well.

For example, spherically symmetric vacuum systems contain no physics, only gauge and constraints. They are already easily unstable.

Idea: Measure the "quality" of a coordinate system. This should lead to a good gauge condition.

This talk will define the term "quality", derive a gauge from that, and then present example simulations.

Interpreting coordinate systems

Two points of view:

A: The metric is a field in coordinate space. The metric evolves like e.g. an electromagnetic quantity. The coordinate space is flat.

B: The coordinate points are located in the physical space. The grid points move about as the physical spacetime is discovered.

Usually A is used to present graphs. A is what is found in codes. A is what is used to define stability.

However, B is much more intuitive. It separates gauge from physics.

Unfortunately, B can only be easily displayed for flat space.

The quality of a coordinate system

Consider the 3-coordinate system during a time evolution. How can one make it stay "good", including at the boundaries?

Criteria: no grid stretching, no grid twisting, no zigzagging, no grid crossing, no falling into the horizon, . . .

In picture B: One has to shepherd the grid points to stay nicely distributed.

Goal: To find a good measure of quality, and to force the gauge to do the Right Thing (TM). Computing time is cheap.

What is a gauge?

Electrodynamics: A gauge removes degrees of freedom

Numerical relativity: Lapse and shift can be chosen freely, independently of the initial data

Lapse and shift define only the time evolution of the gauge: they are a *gauge evolution condition*

What is then a *gauge fixing condition*?

Idea: Define K and $F_i \approx \gamma_{ij,j}$ ahead of time, then choose α and β^i accordingly

The evolution system

Primary quantities similar to the BSSN quantities: ψ, $\tilde{\gamma}_{ij}$, $F_i = \bar{h}_{ij,j}$, K, $\tilde{A}_{ij} = \psi^{-2} A_{ij}$

with the traceless conformal metric
$\bar{h}_{ij} := \tilde{\gamma}_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}\tilde{\gamma}_{kl}$

Specifying K and F_i leads to elliptic eqns. for α and β^i

Enforcing the constraints leads to elliptic eqns. for ψ and a vector potential for $\tilde{A}_{ij}$

Enforcing the gauge on the 3-metric leads to elliptic eqns. for a vector potential for $\tilde{\gamma}_{ij}$

Being able to check and enforce the gauge is the distinctive feature of this system.

Some comments

Again, computing time is cheap. However, waiting does get annoying with time.

Because K and F_i can be freely specified, there are in principle no restrictions on the spacetimes and coordinate systems that can be described.

While specifying K is a straightforward generalisation of maximal slicing, specifying F_i is very different from minimal distortion: MD specifies something equivalent $\partial_t F_i$ only, hence it has no gauge variables to check or enforce.

F_i and $\tilde{\Gamma}^i$ are very similar. Choosing a $\partial_t \tilde{\Gamma}^i = 0$ Gamma freezing shift can be considered equivalent to this system. However, Gamma freezing changes $\tilde{\Gamma}^i$ from time to time (?)

Implementation

Implemented with basic discretisation methods:

- second order centred differencing
- artificial diffusion for advection terms
- second order Runge-Kutta time integrator
- no blending

Regarding the formulation of the equations:

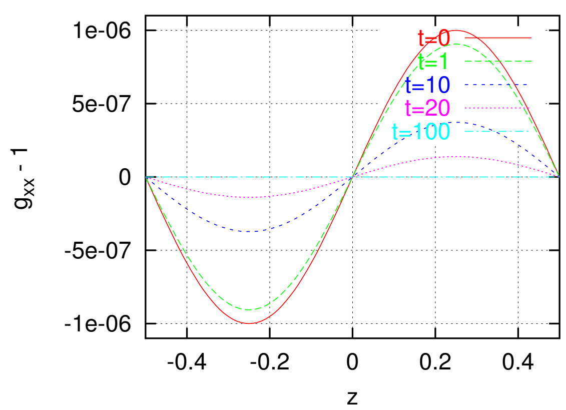
- no rewriting using constraints
- no substitution using F_i for metric derivatives
- no bells and no whistles

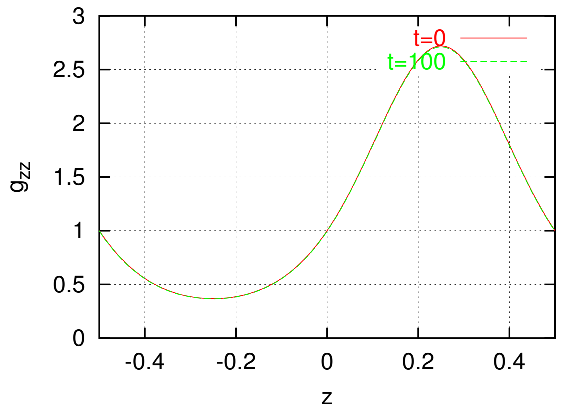
Remark: this system is very stable

Basic tests

weak Bondi wave (linear gravitational wave), gauge wave (nonlinear)

1D domain, Dirichlet boundaries, 100 grid points, artificial diffusion with $C_{\mathsf{SM}} = 1/4$





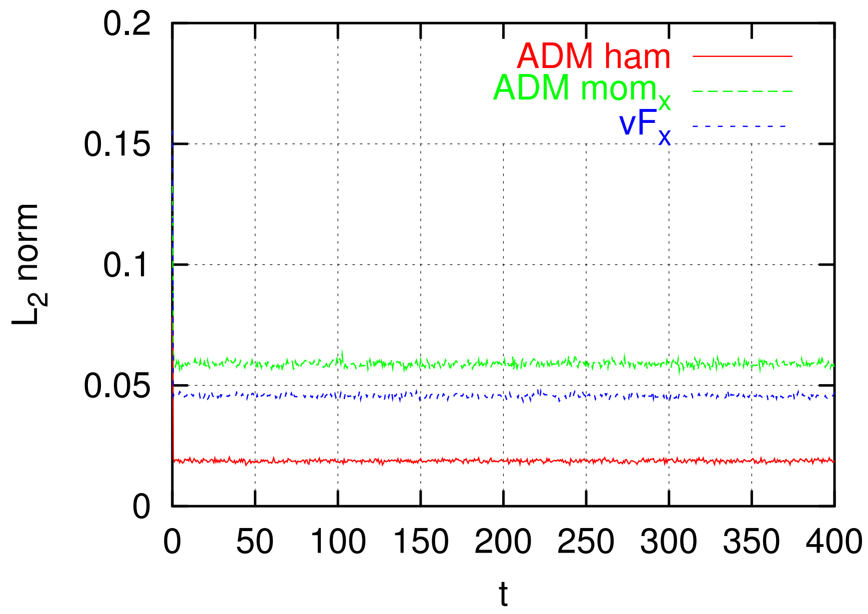
Amplitude decay of the Bondi wave is caused by the artificial diffusion: corresponds to a 10% amplitude loss in a collision run

Stability tests

Szilágyi et al. (PRD **62** 104006 and PRD **65** 064015) define *robust stability* as a numerical test.

Stages I and II require periodicity, which is not well-posed for the elliptic eqns.

The code passes stage III: Dirichlet boundaries on a box, noise on the initial data, noise on the boundaries



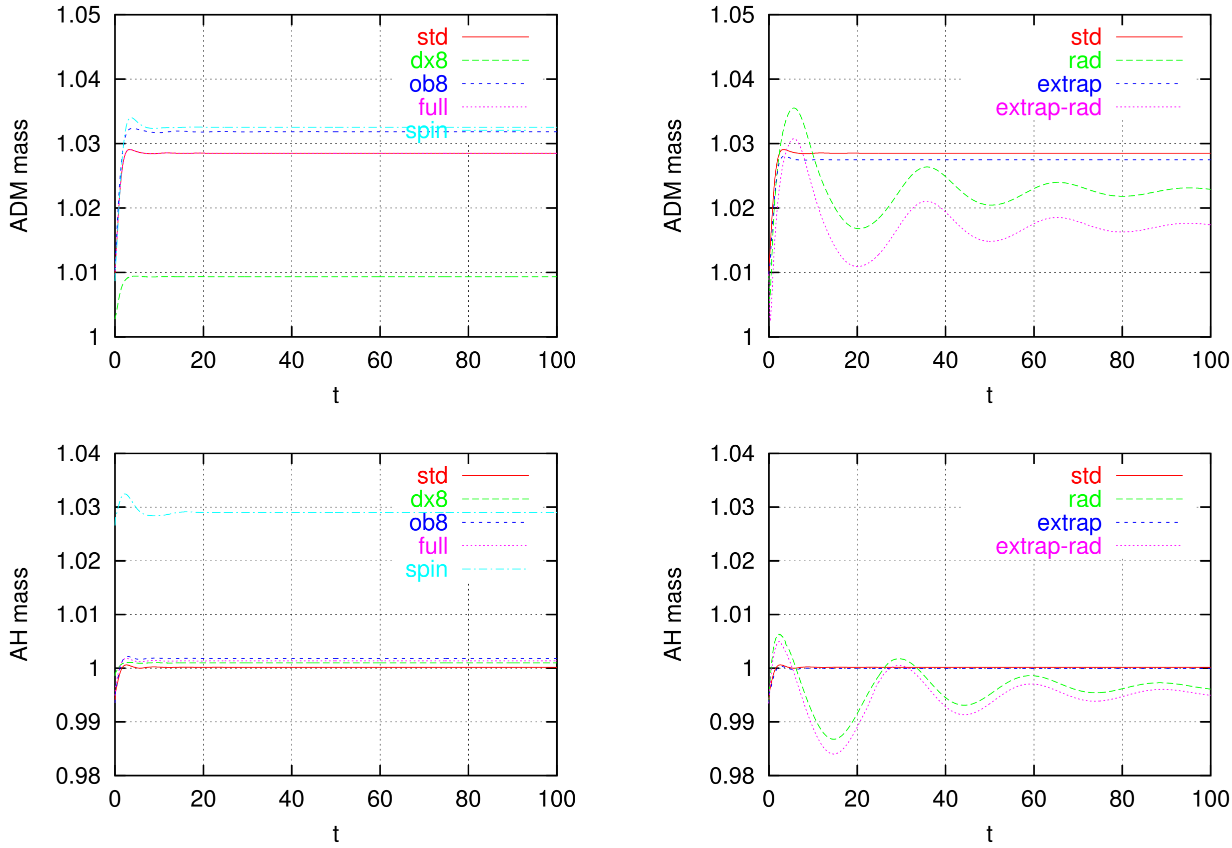
(Note: chose noise level 10^{-1} instead of 10^{-12}, but only 100 instead of 1000 crossing times)

Sleeping black holes

Single Kerr-Schild black time evolution: ADM masses and apparent horizon masses.

std: $M = 1$, $a = 0$, $dx = 1/4$, $x \in [-4; 4]$, octant mode, Dirichlet boundaries

dx8=higher resolution, *ob8*=larger domain, *full*=without symmetries, *spin*=with spin, *rad*=radiative outer boundary, *extrap*=extrapolated inner boundary

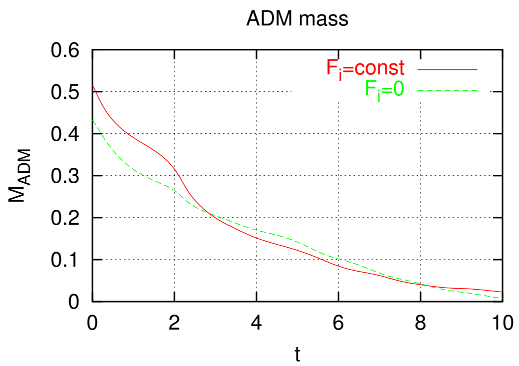


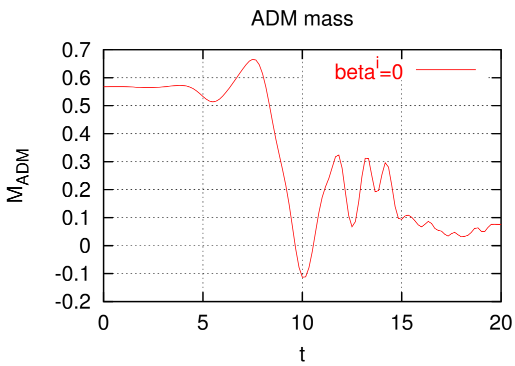
Brill wave

Comparing three runs with different gauge conditions: This system with $F_i = \text{const}$ and with $F_i = 0$, and the AEI BSSN system with $\beta^i = 0$

Amplitude $A = 4$ (subcritical), $K = 0$ (maximal slicing)

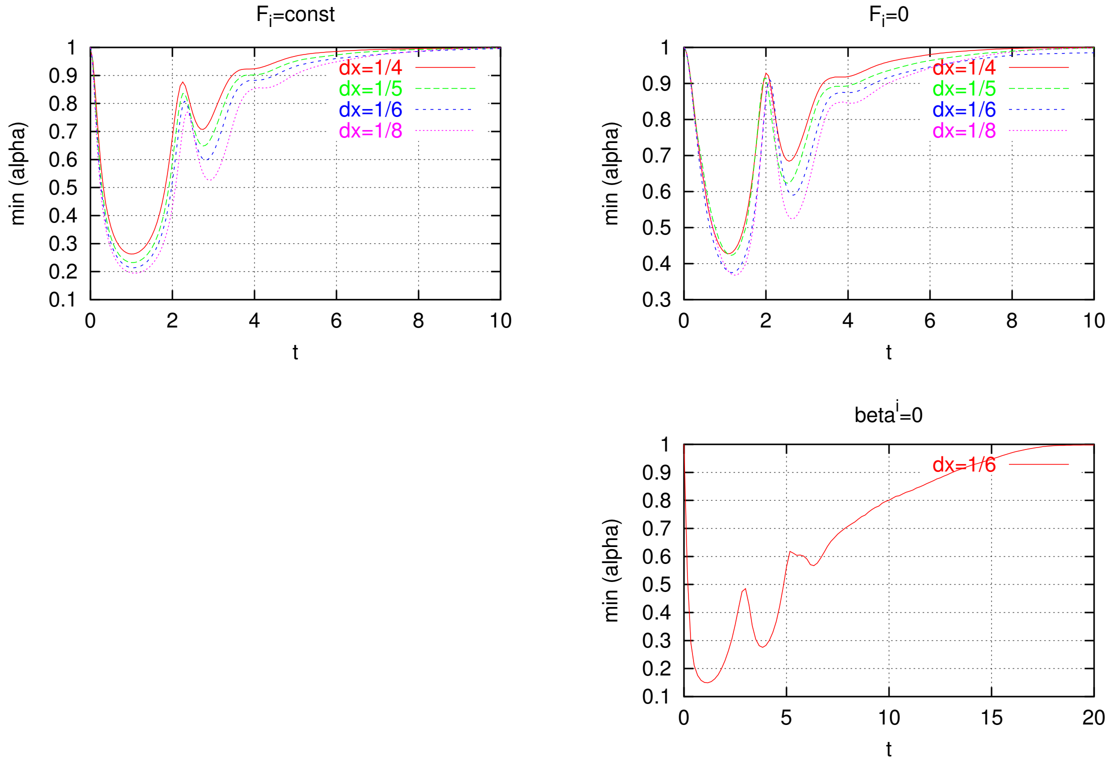
It is difficult to compare different gauges.



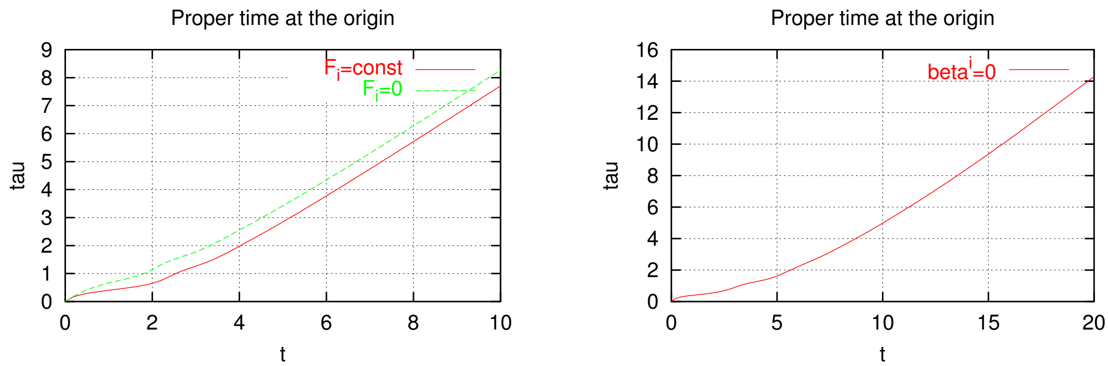


ADM mass vs. time. Note the different time scales

Brill wave (2)



Minimum of the lapse vs. time



Proper time vs. coordinate time at the origin

Conclusion

The intent of *gauge fixing* is choosing a gauge condition that is similar to a constraint, so that it can be monitored and enforced.

Solving elliptic equations can "easily" lead to a stable evolution.

One danger of tools like Cactus is that it is very convenient in places — and things that are not convenient remain undone. We need fast and convenient elliptic solvers.

This system is not the yellow of the egg, but might point into an interesting direction. Look at it as a proof of concept. It should be combined with other methods.