Novel finite-differencing techniques for numerical relativity: application to black hole excision

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Introduction

- Long-term goal: A stable numerical scheme for relativistic physics.
- Define the problem
- Examine system at continuum
- Use discrete analogues of continuum methods
- Scalar field on a curved background

Initial boundary value problem

$$\begin{aligned}
\partial_t u &= A(t,\vec{x})^i \partial_i u + B(t,\vec{x}) u, \\
u(0,\vec{x}) &= f(\vec{x}), \\
w_+(t,\vec{x}) &= S w_-(t,\vec{x}) + g(t,\vec{x}) \; , \vec{x} \in \partial\Omega,
\end{aligned}$$

- An IBVP is well posed if
  - A (smooth) solution exists,
  - The solution is unique,
  - The solution depends continuously on the initial *and* boundary data.
- Numerical stability $\approx$ discrete well-posedness
- Philosophy: Construct “well-posed” FDA

Energy method

- The energy method is a standard test for well-posedness. Define a discrete analogue:

Continuum	Semi-Discrete
$\int_\Omega uv dx$	$(u,v)_\Sigma = \triangle x \sum_i \sigma_i u_i^T v_i$
$\int_a^b v du = uv\|_a^b - \int_a^b u dv$	$(u, Dv)_\Sigma = uv\|_a^b - (v, Du)_\Sigma$
$\mathcal{E} = \int_\Omega u^T H u dx$	$E = (u, Hu)_\Sigma$
$\partial_t \mathcal{E} \leq \cdots$	$\partial_t E \leq \cdots$

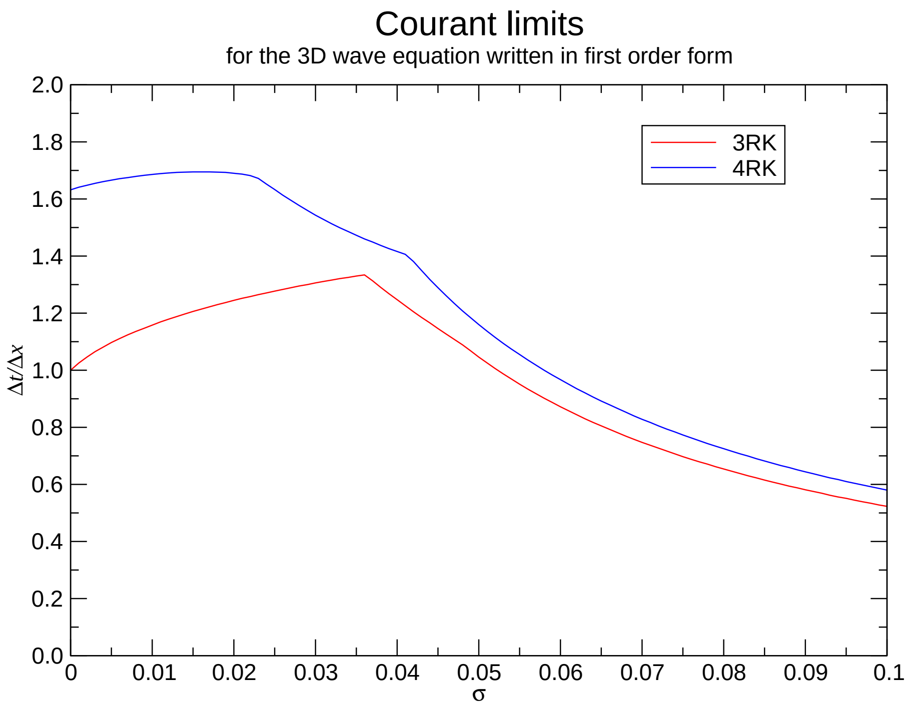
- *Summation by parts* (line 2) is a key condition for semi-discrete systems. Σ and D are chosen to satisfy SBP.

Boundary conditions

- We may only specify ingoing modes for a well-posed IBVP.
- Generic applications of BCs do not preserve SBP.
- We use Olsson's technique for projecting the RHS of the semi-discrete system to construct BCs..
- Maximally dissipative BCs are used for the incoming modes.
- Edge and corner boundary points make a finite contribution to the discrete energy estimate. They must be handled carefully!

Fully discrete system

- Use MOL to solve semi-discrete system
- Time integrator must preserve the energy estimate
- RK3 and RK4 meet this condition.



The hyper code

- A code infrastructure for general hyperbolic systems.
- Based on Cactus Computational Toolkit (I/O, MPI, etc.)
- Code is modular, and data structures imitate notation for vector valued functions.

$$\partial_t u = A(t, \vec{x})^i \partial_i u + B(t, \vec{x}) u$$

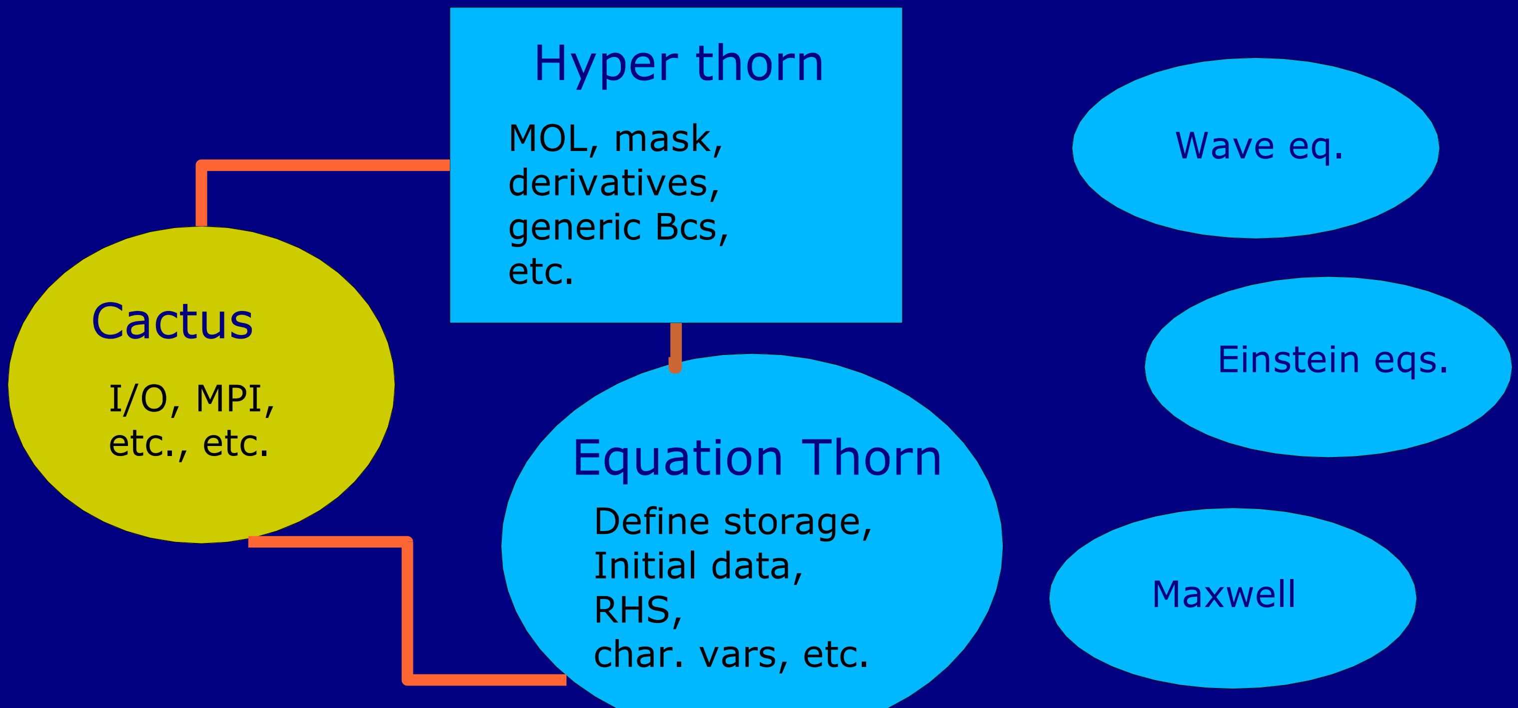
- In C:

  `double **u;`

  `u[`*function index*`][`*spatial index*`];`

- Only the $!# % thing is written in F90. . .

hyper code in cartoon form



Wave equation

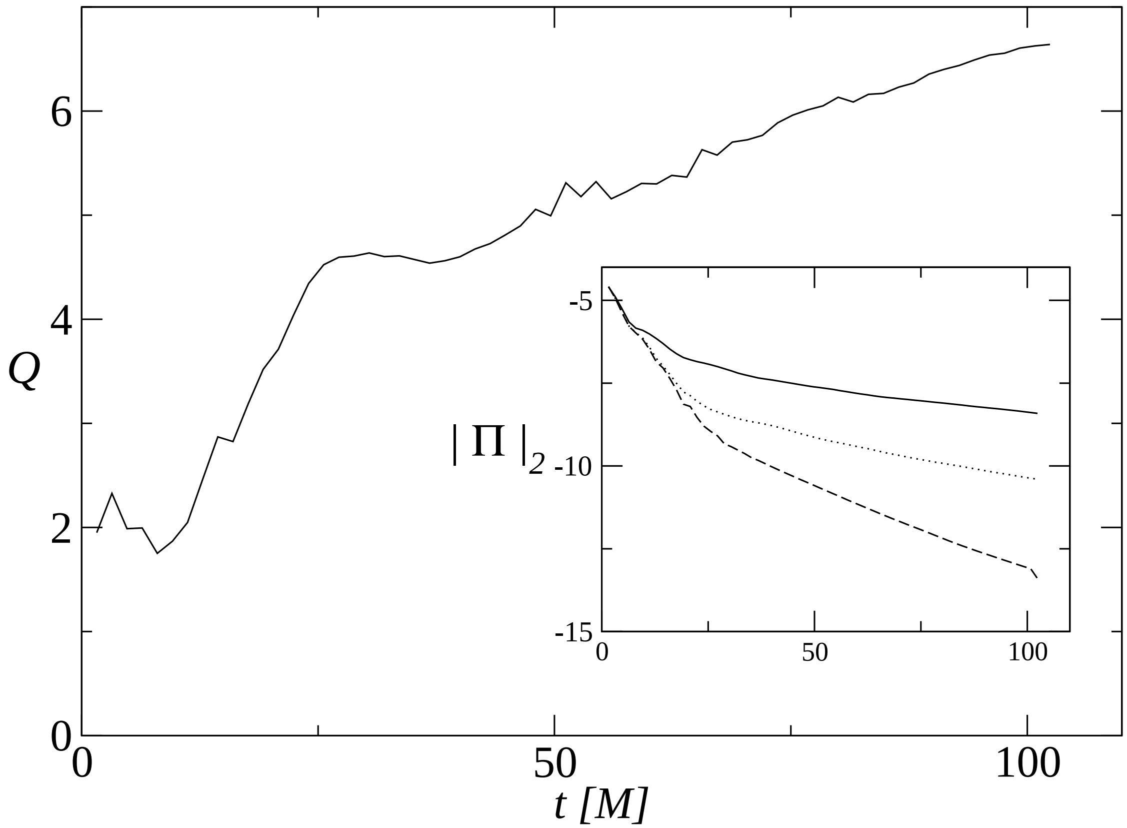
- Klein-Gordon field on a Schwarzschild background
  - Kerr-Schild & Painlevé-Gullstrand
- Write equation of motion as first order system in an *interpolating formulation*
  - globally symmetric hyperbolic
  - constraint violating modes do not enter the domain through outer boundary

$$\pounds_q \Phi = u^\mu V_\mu \equiv \Pi, \quad \nabla^\mu V_\mu = 0, \quad \pounds_q V_\mu = \nabla_\mu \Pi,$$

$$q^i = (k^i - b^i)/\alpha, \quad b^i = f(r)\beta^i$$

$f(r) = 1$ near the EH, and varies smoothly to zero at OB

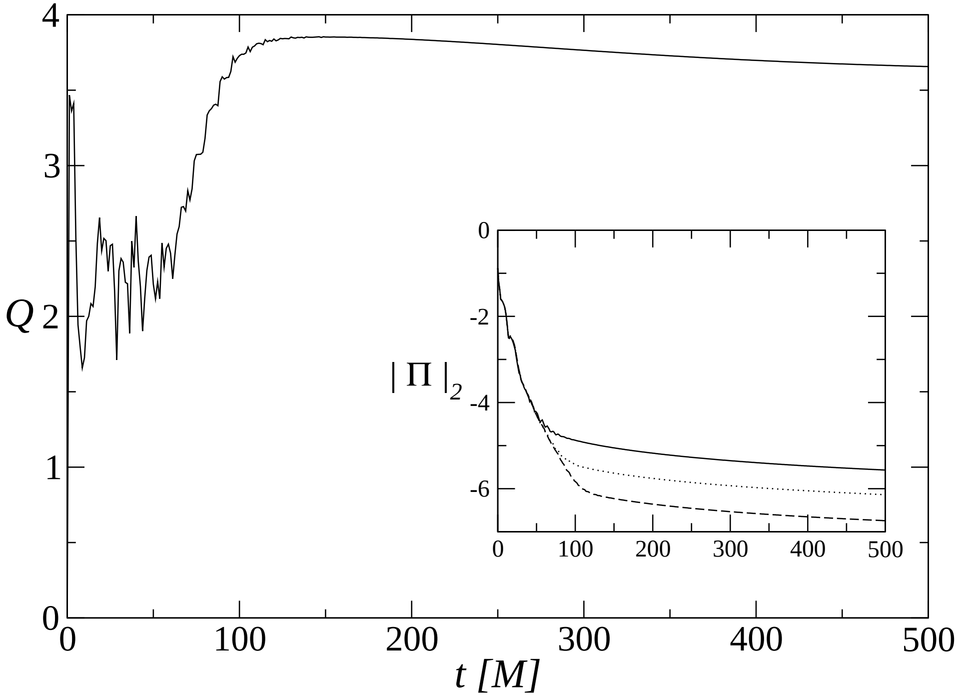
Wave equation: self convergence



Grid domain: $x^i \in [1.5, 5.5]$, CFL = 0.8, Points: 41^3, 81^3 161^3. See movies at

http://relativity.phys.lsu.edu/movies/scalarfie

Excised Wave: self convergence



Grid domain: $x^i \in [-4, 4]$, excised region $x^i \in [-0.375, 0.375]$, CFL = 1.0, $\sigma_{\rm diss} = 0.02$, Points: 65^3, 129^3 257^3.

Conclusion

- Power tools for FDAs
  - Well-posed IBVP at continuum
  - Properly defined D_i and Σ (also Q)
  - Careful attention to BCs
  - Stability conditions for time integration
- Future directions
  - Moving boundaries (boosted black holes)
  - Spherical excision
  - Einstein equations and other applications (currently 6 equation thorns active in CVS)