Detecting ill posed boundary conditions in General Relativity

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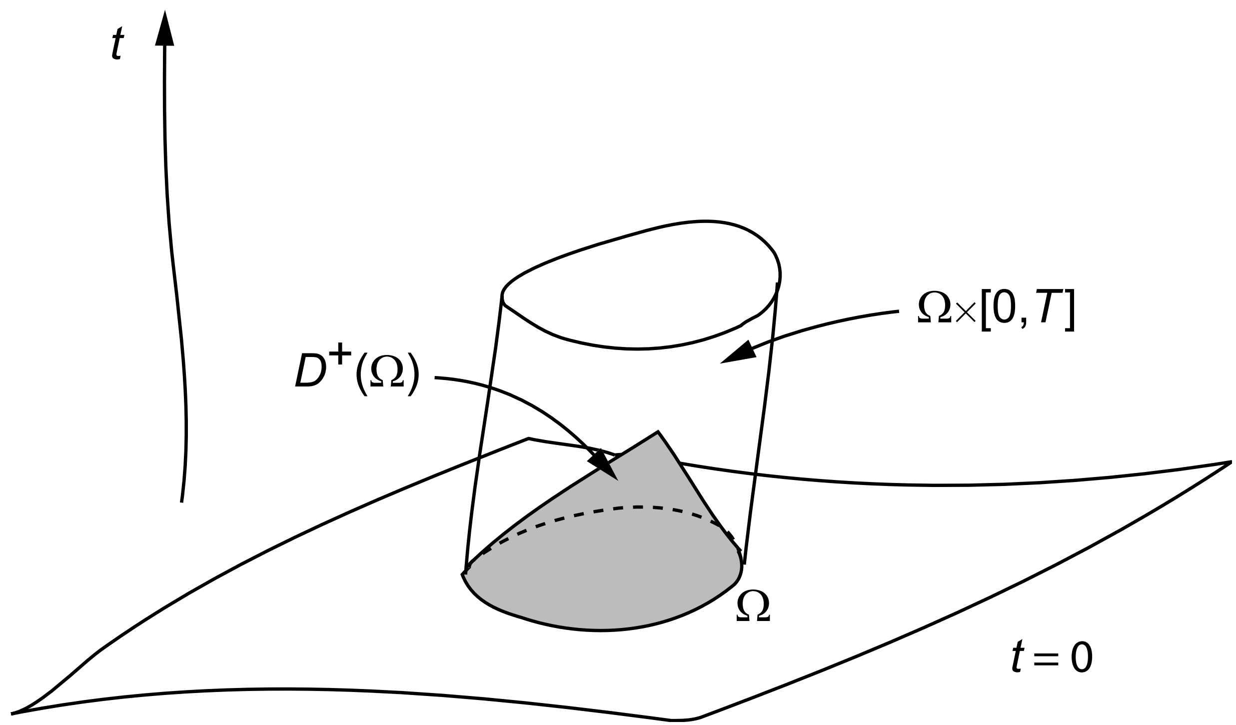
The plan

- Introduction
  - Initial-boundary value problem in General Relativity.
  - Well posedness.
- Laplace-Fourier technique
  - Useful tool for the detection of ill posed modes.
  - Generalized Einstein-Christoffel (EC) system linearized around Minkowski with different types of boundary conditions.
- Conclusions

Initial-boundary value problem



- How can we give boundary data on $\partial\Omega \times [0, T]$ so that the constraints vanish inside the cylinder $\Omega \times [0, T]$?

Well posedness

- The initial-boundary value problem

$$\partial_t u = A\partial_x u + B\partial_y u \qquad t > 0, x > 0$$
$$u(0,x,y) = f(x,y)$$
$$Lu(t,0,y) = g(t,y)$$

is said to be *well posed* if for all smooth compatible data there is a unique smooth solution $u(t,x,y)$ and, in any finite time interval $0 \le t \le T$ the solution can be estimated in terms of the data

$$\|u(t,\cdot)\|^2 \le K_T \left[\|f\|^2 + \int_0^t \|g(\tau,\cdot)\|^2 d\tau \right]$$

- Continuous dependence on the data $\Leftarrow$ construction of stable (and consistent) finite difference approximations.

Common techniques

- The energy method. Symmetrizable hyperbolic systems with maximal dissipative boundary conditions

$$w^{(-)}(t,0,y) = S(t,y)w^{(+)}(t,0,y) + g(t,y)$$

  Gives sufficient conditions for well posedness.

- The Laplace-Fourier transform method. More general types of boundary conditions

$$L(\partial_t, \partial_y)u = g(t,y)$$

  Often gives necessary and sufficient conditions for well posedness.

- Both techniques have a numerical counterpart.

Previous work

- Maximal dissipative boundary conditions
  - Friedrich & Nagy (1998): full nonlinear vacuum equations, based on Weyl-like formulation with tetrads.
  - Szilagyi & Winicour (2002): nonlinear vacuum equations in harmonic coordinates, only homogeneous boundary data.
  - CPSTR (2002): linearization around a Minkowski background.
- More general types of boundary conditions
  - Stewart (1998): Frittelli-Reula system linearized about Minkowski.
  - CLT (2001): numerical experiments with non linear EC system in spherical symmetry.
  - Frittelli & Gómez (2003): projection of Einstein equation along the normal to the boundary surface.

How to detect ill posed modes

- Look for solutions of the homogeneous ($g = 0$) boundary value problem of the form

$$u(t,x,y) = e^{st+i\omega y}\tilde{u}(x) \tag{1}$$

where $s \in \mathbb{C}$ with $\Re(s) > 0$ and $\tilde{u} \in C^\infty \cap L_2(0,\infty)$.

- If such solution exists then the norm of the solution $u(t,\cdot)$ cannot be bounded in terms of the data: the problem cannot be well posed.

$$u_m(t,x,y) = e^{m(st+i\omega y)}\tilde{u}(mx)\ , \qquad \frac{\|u_m(t,\cdot)\|}{\|u_m(0,\cdot)\|} = e^{m\Re(s)t}$$

How to detect ill posed modes

- Inserting (1) into the evolution equations and boundary conditions leads to a family of ordinary boundary value problems on the half-line $x \geq 0$. The zero speed modes can be eliminated and new variables $v_\pm(x)$ can be introduced.

$$\partial_x v_-(x) = M_-(s,\omega)v_-(x) + M_0(s,\omega)v_+(x) \ ,$$
$$\partial_x v_+(x) = M_+(s,\omega)v_+(x) \ ,$$
$$L_- v_-(0) + L_+ v_+(0) = 0,$$

  $M_+(M_-)$ is upper triangular and its eigenvalues have positive (negative) real part.

- Impose the boundary conditions at $x = \infty$ ($\int_0^\infty |\tilde{u}(x)|^2 dx < \infty$) $\Rightarrow$ $v_+(x) = 0, v_-(x) = e^{M_- x}\sigma_-$, with $L_-\sigma_- = 0$.

How to detect ill posed modes

- If $L_-(s,\omega)\sigma_- = 0$ has a non trivial solution with $\Re(s) > 0$, then the initial-boundary value problem is *ill posed* in any sense.
- Remarks:
  - With this technique one can quickly rule out ill posed BC.
  - It can be used to analyze BC that have the form of a partial differential equation at the boundary.
  - Frozen coefficient principle: if all frozen-coefficient problems are well-posed then the variable coefficient problem is also well-posed.

Linearized gEC system

- 6 uncoupled wave equations (written in FOSH form) whose solution has to satisfy 22 constraints. One free parameter η. For $\eta = 4$ one recovers the Einstein-Christoffel system.
- Half-space problem ($x \geq 0$) with BC at $x = 0$ of the form

$$
\begin{aligned}
V_x^{(-)} - aV_x^{(+)} &= 0, && \left(\partial_t(u_{BB}^{(-)} + au_{BB}^{(+)}) = \ldots\right), \\
V_A^{(-)} - bV_A^{(+)} &= 0, && \left(\partial_t(u_{xA}^{(-)} + bu_{xA}^{(+)}) = \ldots\right), \\
u_{xx}^{(-)} - c\, u_{xx}^{(+)} &= g_{xx}, \\
\hat{u}_{AB}^{(-)} - d\, \hat{u}_{AB}^{(+)} &= \hat{g}_{AB},
\end{aligned}
$$

where $V_j^{(\mp)}$ denotes the constraint characteristic field and $u^{(\mp)}$ the characteristic field for the main evolution system. Furthermore, $|a|, |b|, |c|, |d| \leq 1$.

Results

- In this case the determinant of L_- is given by

$$(6+4\zeta^2)\left[(1+a\psi(\zeta))(1+b\psi(\zeta)) - \left(1-\frac{3}{4}\eta\right)^2 (1+a)(1+b)\psi(\zeta)\right],$$

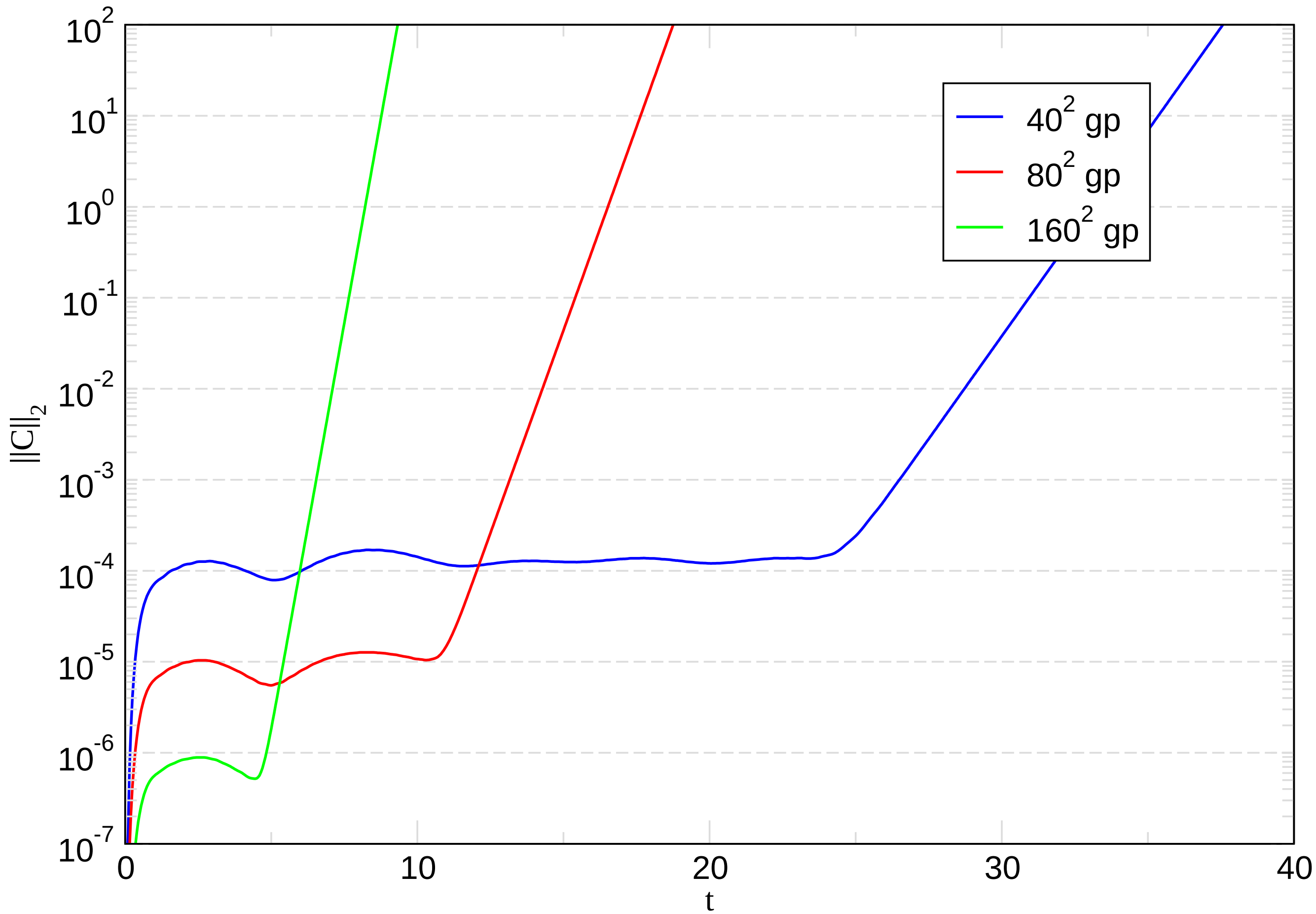
  where $\zeta = s/|\underline{\omega}|$ and $|\psi(\zeta)| < 1$ for $\Re(\zeta) > 0$.
- $a = \mp 1, b = c = d = \pm 1$: Neumann and Dirichlet BC of CPSTR; it has no ill posed modes.
- $a = b = 0$: there are ill posed modes iff $\eta < 0$ or $\eta > 8/3$.
- $a = 0$, $b = 1$: equivalent to $G_{xy} = G_{xz} = G_{xx} - G_{xt} = 0$; there are ill posed modes iff $\eta < 0$ or $\eta > 8/3$.
- Remark: these ill posed modes are constraint violating.

Constraint violating modes

- The ill posed modes found are all *constraint violating* modes.
- Main system: symmetric hyperbolic for any $\eta \neq 0$, knows about constraints through BC (not maximal dissipative).
- Evolution of the constraints: strongly hyperbolic for any $\eta \neq 0$, symmetric hyperbolic for $0 < \eta < 2$, maximal dissipative BC.
  - Applying maximal dissipative BC to a strongly hyperbolic system (*not* symmetrizable) can lead to an ill posed problem: e.g. evolution of the constraints for $\eta < 0$ or $\eta > 8/3$ with boundary conditions $V^{(-)}(t,0,y) = 0$.

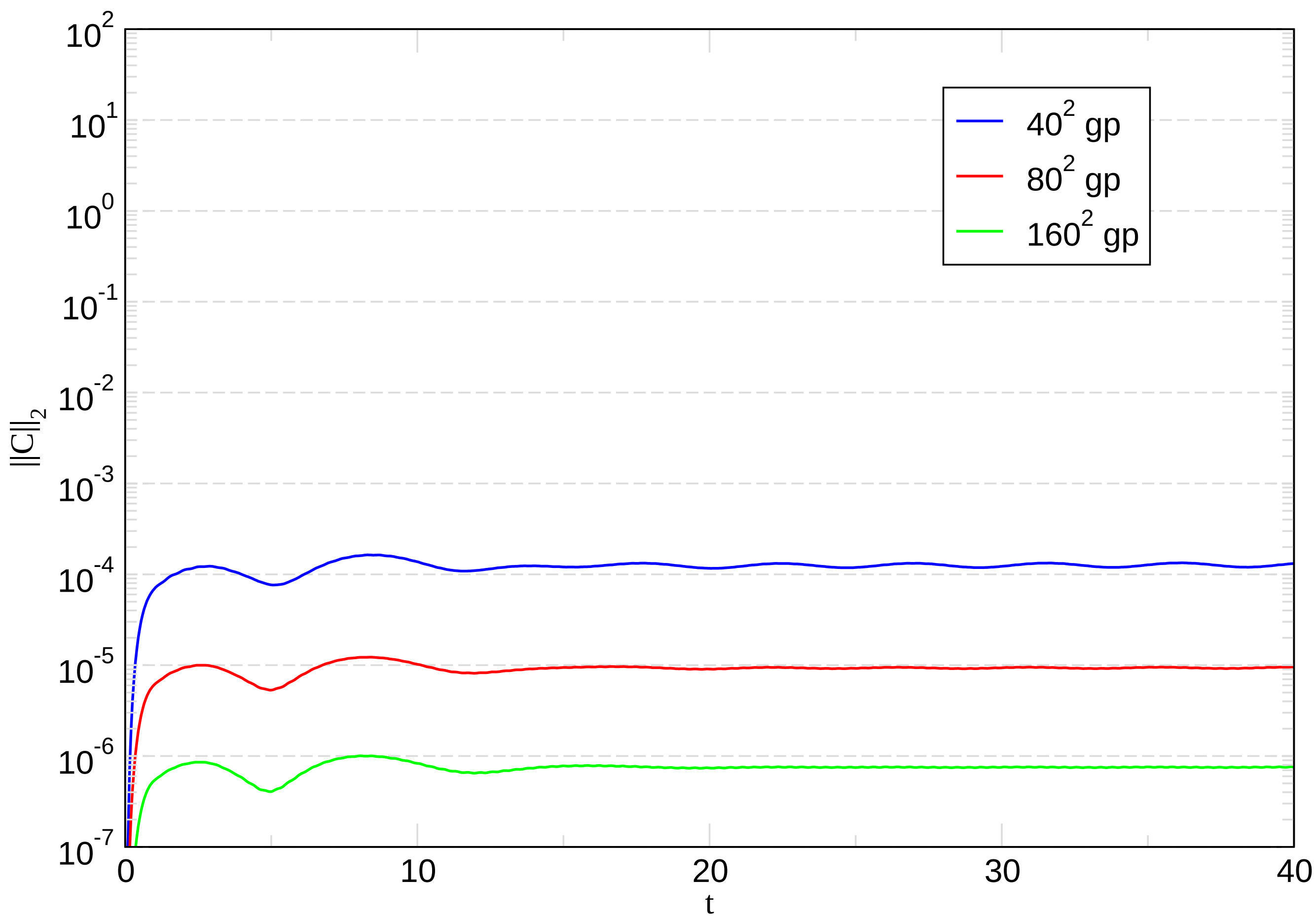
Numerical evidence

$V_j^{(-)} = 0 \qquad (\eta = 2.7)$



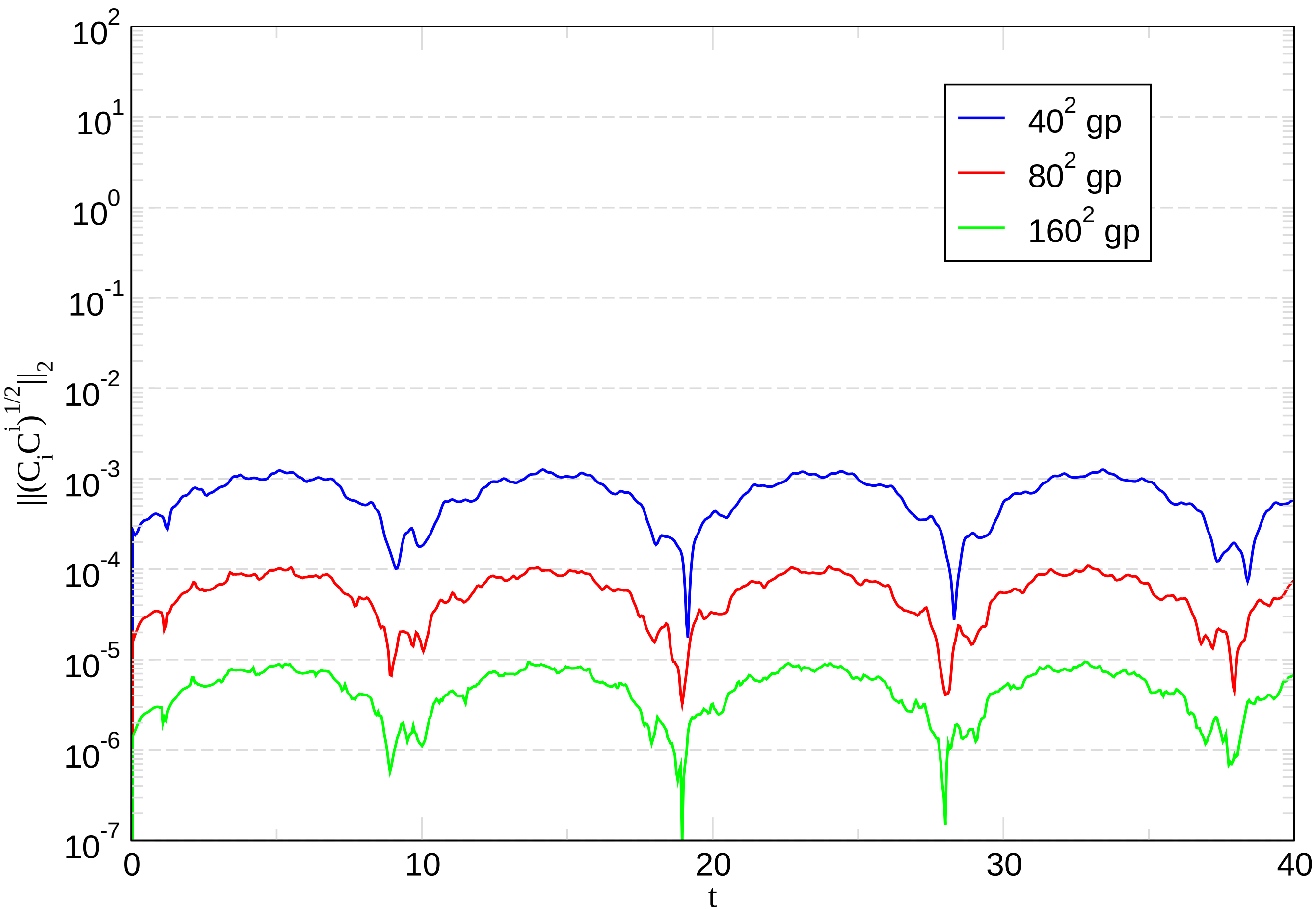
Numerical evidence

$V_j^{(-)} = 0 \qquad (\eta = 2.6)$



Numerical evidence

$$V_x^{(-)} - V_x^{(+)} = V_A^{(-)} + V_A^{(+)} = 0 \qquad (\eta = 1.0)$$



Conclusions

- Extra care is needed when boundary conditions are given in differential form.
- Ill posed boundary conditions do not preserve the constraints.
- It is important to look at the evolution system for the constraints.
- Numerical evidence confirms analytical results.
- Destructive part is done. Constructive part (proving well-posedness when there no ill posed modes) is next.