Exponential Stretch-Rotation (ESR) transformation in GR

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Summary of two papers:

- **A scalar hyperbolic equation with GR-type non-linearity**, gr-qc/0303063
- **Exponential stretch-rotation formulation of Einstein's equations**, gr-qc/0303065.

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1 A scalar non-linear hyperbolic PDE

Consider

$$g_{tt} = g_{xx} - g^{-1}(\alpha g_t^2 + \beta g_x^2 + \gamma g_x g_t), \tag{1}$$

inspired by

$$R_{ab} = 0, \quad R \sim \sum \partial\Gamma + \sum \Gamma\Gamma, \quad \Gamma \sim g^{-1}\partial g. \tag{2}$$

Can also be written as

$$\begin{aligned} &g_t = K, \\ &K_t = g_{xx} - g^{-1}(\alpha K^2 + \beta g_x^2 + \gamma K g_x) \end{aligned} \tag{3}$$

or as

$$\begin{aligned} g_t &= K, \\ K_t - D_x &= R, \\ D_t - K_x &= 0, \end{aligned} \tag{4}$$

where

$$R \equiv -g^{-1}(\alpha K^2 + \beta D^2 + \gamma D K), \quad K \equiv g_t, \quad D \equiv g_x. \tag{5}$$

The equation has a non-linearity similar to that of the equations of GR. Integration of this equation poses a problem. One can have a numerical stability and convergence at any fixed moment of time. Yet, a long-term integration is asymptotically unstable.

An example:

Consider this analytic solution

$$g = \left(x + \frac{t}{10}\right)^{-3.88}. \tag{6}$$

Use a second-order accurate explicit predictor-corrector scheme

$$\text{CFLN1}: \quad \begin{cases} \bar{K}_i = K_i^{n-\frac{1}{2}} + \Delta t \left(\dfrac{g_{i+1}^n - 2g_i^n + g_{i-1}^n}{\Delta x^2} + \mathcal{R}(g_i^n, K_i^{n-\frac{1}{2}}, D_i^n) \right) \quad \text{(predictor)}, \\ K_i^{n+\frac{1}{2}} = \bar{K}_i + \dfrac{\Delta t}{2} \left(\mathcal{R}(g_i^n, \bar{K}_i, D_i^n) - \mathcal{R}(g_i^n, K_i^{n-\frac{1}{2}}, D_i^n) \right) \quad \text{(corrector)}, \\ g_i^{n+1} = g_i^n + \Delta t \, K_i^{n+\frac{1}{2}}, \end{cases} \tag{7}$$

or a second-order accurate method-of-lines scheme

$$\text{MOL1(n)}: \quad \begin{cases} \dfrac{\partial g_i}{\partial t} = K_i, \\ \dfrac{\partial K_i}{\partial t} = \dfrac{g_{i+1} + g_{i-1} - 2g_i}{\Delta x^2} + \mathcal{R}(g_i, K_i, D_i), \end{cases} \tag{8}$$

combined with a RK or an ICN time-integrator.

Obtain the solution numerically on interval $0.1 \le x \le 1.1$, and $t > 0$ using N grid points:

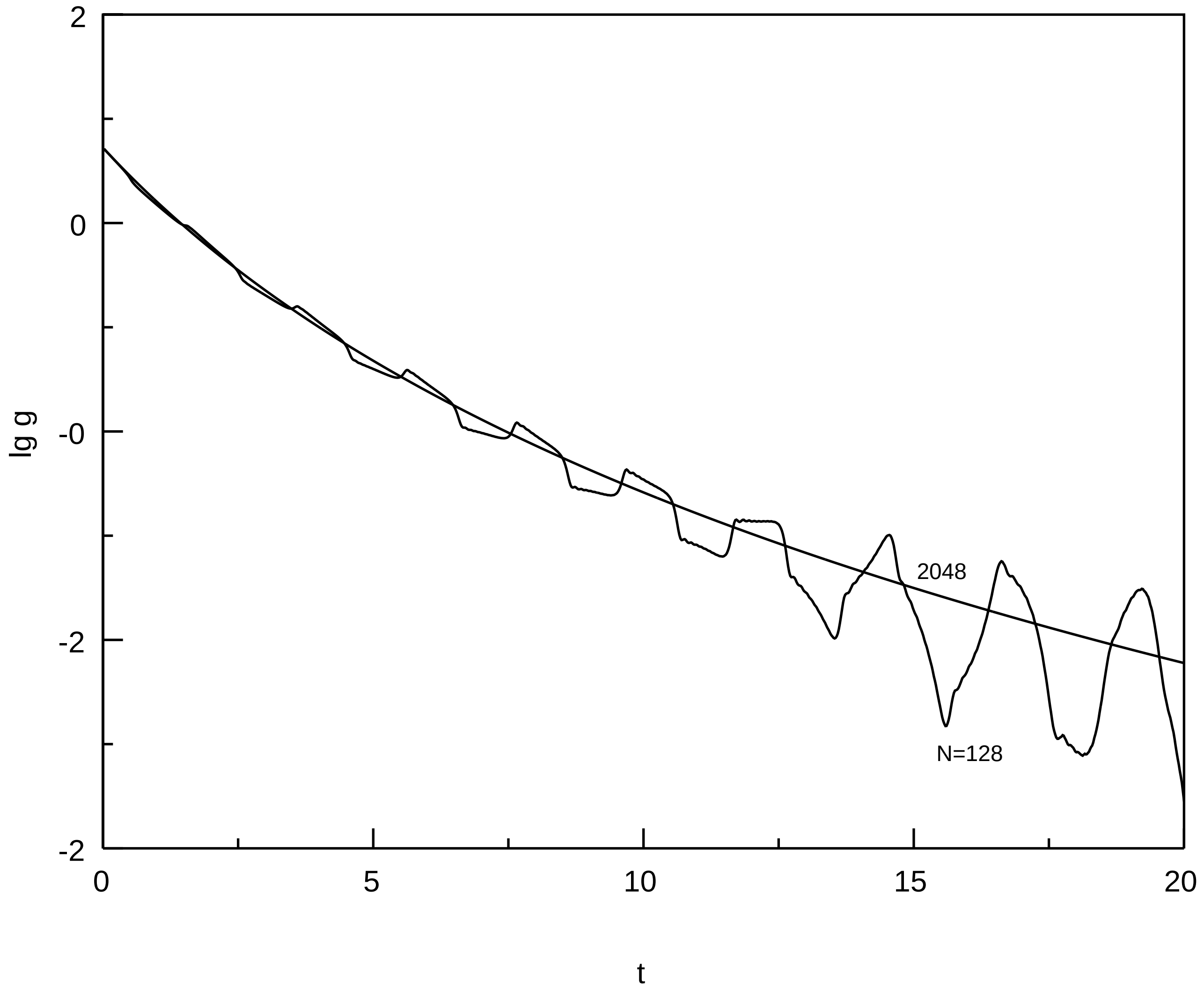
Convergence at fixed t:

N	$L_\infty(t_1)$	$L_\infty(t_2)$
64	2.9E-01	NaN
128	1.3E-01	2.2E-01
256	3.0E-02	8.1E-02
512	7.5E-03	1.9E-02
1024	1.9E-03	4.8E-03
2048	4.7E-04	1.2E-03

Asymptotic instability at $t \to \infty$:

Figure on the next page compares numerical solutions in the middle of the interval to the exact solution and illustrates the "instability." **Convergence is not uniform. Convergence at late times requires exponentially large computational resources,**

$$\Delta x \sim e^{-t}. \tag{9}$$


2
0
-0
-2
-2
lg g
0
5
10
15
20
t
2048
N=128

Long-term integration can be improved by an exponential transformation:

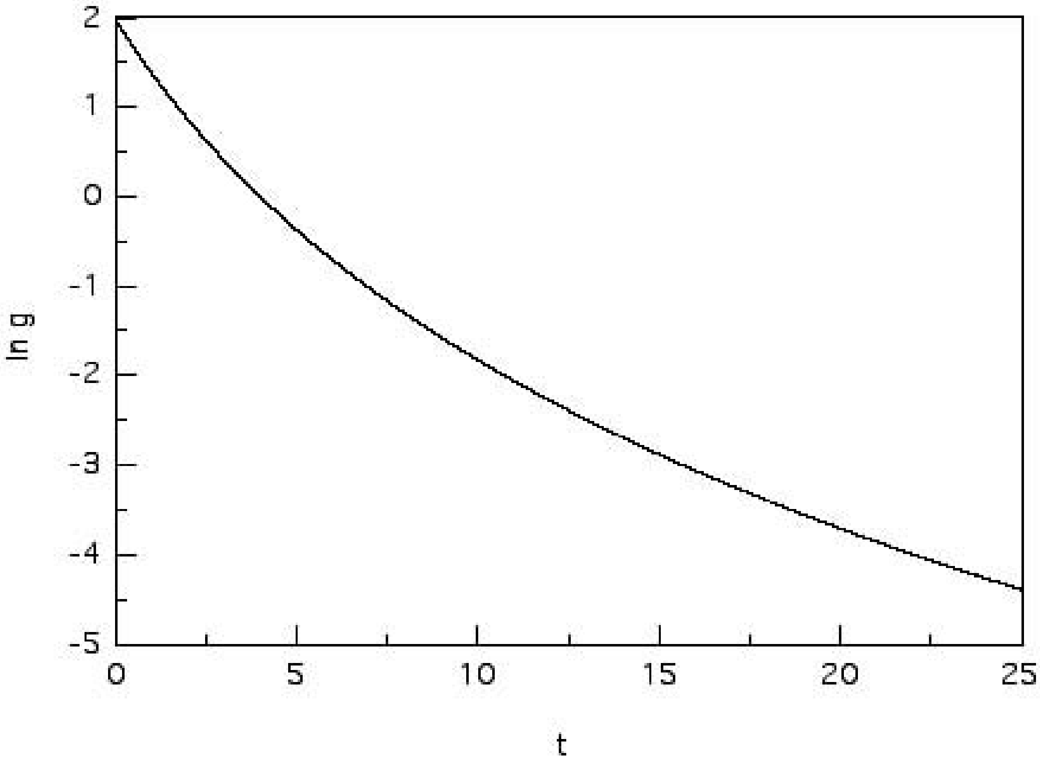
$$g = e^{\phi}, \tag{10}$$

where ϕ is a new unknown. The transformed equation is

$$\phi_{tt} = \phi_{xx} - (\alpha + 1)\phi_t^2 - (\beta - 1)\phi_x^2 - \gamma\phi_x\phi_t. \tag{11}$$

The transformation (10) removes g^{-1} multiplier in front of the non-linear term in (1), and maps $0 < g < \infty$ onto $-\infty < \phi < \infty$ so that values of $g \leq 0$ are excluded.

Solutions obtained in logarithmic variables using CFLN1 scheme with $cfl = 1$ are shown on the next figure. Solid lines - numerical solutions for $N = 128$ and $N = 2048$. Dashed line - exact solution. The solutions cannot be distinguished on the plot.


2
1
0
-1
-2
-3
-4
-5
ln g
0
5
10
15
20
25
t

2 Exponential stretch-rotation transformation in GR

A similar transformation is possible in GR. Write a three-dimensional metric of space-like hypersurfaces as

$$\gamma_{ij} = A^{\dagger}_{ik} D_{kl} A_{lj} = e^{\epsilon_{ikm}\theta_m} e^{\phi_k} e^{-\epsilon_{jkn}\theta_n}, \tag{12}$$

where $D_{ij} = \delta_{ij}\lambda_i$ is a diagonal matrix of eigenvalues of γ_{ij}, and A_{ij} is the orthogonal matrix of rotations, $A^{\dagger}_{ij} = A^{-1}_{ij}$; superscript † denotes a matrix transposition, $A^{\dagger}_{ij} = A_{ji}$, and $A_{mi}A_{mj} = \delta_{ij}$, ϕ_k are logarithms of the eigenvalues of γ_{ij}, θ_k are rotation angles. Decomposition (12) is always possible for a symmetric matrix. It leads to a "unique" formulation of GR in terms of ESR variables.

Steps in re-writing GR equations in terms of ESR variables:

1. Use Rodrigues formula to explicitly write A_{ij} in terms of ϕ_k and θ_k.

2. Write differentials of the metric in terms of differentials of ϕ_k and θ_k.

$$d\gamma_{ij} = A_{ki}A_{kj}e^{\phi_k}d\phi_k + A_{ni}A_{mj}C_{k;nm}d\theta_k, \tag{13}$$

where

$$C_{k;ij} \equiv A_{in}B^{\dagger}_{k;nm}D_{mj} + D_{in}B_{k;nm}A^{\dagger}_{mj} = A_{in}B_{k;jn}e^{\phi_j} + A_{jn}B_{k;in}e^{\phi_i}, \tag{14}$$

where

$$B_{k;ij} \equiv \frac{\partial A_{ij}}{\partial \theta_k} \tag{15}$$

3. Invert (13) **to write $d\phi_k$ and $d\theta_k$ in terms of $d\gamma_{ij}$.**

$$d\phi_i = e^{-\phi_i}A_{im}A_{in}d\gamma_{mn}. \tag{16}$$

$$d\theta_i = C^{-1}_{ij}|\epsilon_{jmn}|A_{mr}A_{nk}d\gamma_{rk}, \tag{17}$$

where

$$C_{ij} = |\epsilon_{inm}| C_{j;nm}, \tag{18}$$

4. Similarly, express second differentials $d^2\phi_k$ and $d^2\theta_k$ in terms of $d^2\gamma_{ij}$ and $d\gamma_{ij}$.

5. Obtain expressions for ${}^{(3)}\Gamma^i_{jk}$ and Ricci tensor ${}^{(3)}R_{ij}$.

In new variables, γ^{ij} multiplicators disappear from Ricci tensor. Left are terms proprtional to

$$\text{Ricci} \propto \sum e^{\phi_i - \phi_j} \times (\text{partial derivatives of ESR variables}). \tag{19}$$

6. Start with an ADM formulation. Use the above results to rewrite ADM in terms of new variables.

$$\frac{\partial \phi_k}{\partial t} = \psi_k, \quad \frac{\partial \theta_k}{\partial t} = \omega_k, \tag{20}$$

$$\begin{aligned} \frac{\partial \psi_i}{\partial t} - \frac{\partial \ln \alpha}{\partial t}\psi_i =& 2\alpha^2 e^{-\phi_i} A_{in} A_{im} \left(\frac{1}{\alpha} \nabla_n \nabla_m \alpha - R_{nm} \right) \\ &+ (\text{terms quadratic in } \psi_i \ , \theta_i) \\ &+ (\text{shift dependent terms}) \end{aligned} \tag{21}$$

$$\begin{aligned} &\frac{\partial \omega_k}{\partial t} - \frac{\partial \ln \alpha}{\partial t}\omega_k + \frac{\omega_k}{2} \sum \psi_i = \\ &C^{-1}_{kr} |\epsilon_{rij}| \Big(2\alpha A_{in} A_{jm} \nabla_n \nabla_m \alpha - 2\alpha^2 A_{in} A_{jm} R_{nm} + (\text{quadratic terms...}) \Big). \end{aligned} \tag{22}$$

Equations (21), (22) **together with two equations** (20) **constitute an evolution part of an ADM system transformed to ESR variables.**

Evolution part of Einstein's equations, formulated in terms of ϕ_k and θ_k, describes time evolution of the metric at every point of a

hyper-surface as a continuous stretch and rotation of a local coordinate system in a tangential space.

ESR formulation presented above can be modified and extended by introducing new variables such as spatial derivatives of the metric, and by addition of various combination of constraints, similar to modifications previously introduced to a standard ADM formulation.

We are in the process of exploring the utility of ESR for numerical integration of GR equations (Hansen et al., in preparation).