

Exponential stretch-rotation formulation of Einstein's equations

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Abstract

We study a tensorial exponential transformation of a three-dimensional metric of space-like hypersurfaces embedded in a four-dimensional space-time, $\gamma_{ij} = e^{\epsilon_{ikm}\theta_m} e^{\phi_k} e^{-\epsilon_{jkn}\theta_n}$, where ϕ_k are logarithms of the eigenvalues of γ_{ij} , θ_k are rotation angles, and ϵ_{ijk} is a fully anti-symmetric symbol. Evolution part of Einstein's equations, formulated in terms of ϕ_k and θ_k , describes time evolution of the metric at every point of a hyper-surface as a continuous stretch and rotation of a local coordinate system in a tangential space. The exponential stretch-rotation (ESR) transformation generalizes particular exponential transformations used previously in cases of spatial symmetry. The ESR 3+1 formulation of Einstein's equations may have certain advantages for long-term stable integration of these equations.

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1 Introduction

3+1 formulations of general relativity (GR), commonly used in numerical integration, break Einstein's equations onto constraints that must be solved to obtain initial conditions on an initial space-like three-dimensional hypersurface, and a Cauchy part, an evolution problem that is then integrated forward in time [1, 2]. It is well known that integration in time is often unstable and terminates prematurely. In particular, this happens frequently in numerical calculations of black hole collisions which are expected to be a major source of gravitational radiation gravitational observatories [3]. There is no complete understanding of the causes of instabilities. Gauge and constraints instabilities of Einstein's equations are definitely a part of the problem [4]. Bad choice of a numerical scheme may be another. Einstein's equations are highly non-linear, and finding a numerical method capable of stable long-term integration of these equations remains an outstanding problem in numerical GR.

In [5], we gave an example of an asymptotic non-linear instability which arises in a simple scalar hyperbolic equation, $g_{tt} = g_{xx} - \frac{1}{g} (\alpha g_t^2 + \beta g_x^2 + \gamma g_x g_t)$, whose non-linear term mimics a quadratic non-linearity of much more complex Einstein's equations. This equation is well-posed and it does not suffer from gauge or constraint instabilities. Stable numerical schemes applied to this equation converge to exact solutions at any fixed time t when numerical resolution is increased. However, numerical integration can become asymptotically unstable if the resolution is kept fixed and t is increased, $t \rightarrow \infty$. The instability is caused by breaking down of a delicate balance between linear and non-linear terms in the discretized versions of the equation. An important point is that the breakdown takes place without violating second-order accuracy of a discretization. Therefore, having both convergence at finite t and an asymptotic instability at $t \rightarrow \infty$ is not a contradiction.

We studied in [5] an exponential transformation $g = e^\phi$, which maps $0 < g < \infty$ onto $-\infty < \phi < \infty$ and leads to a modified equation $\phi_{tt} = \phi_{xx} - (\alpha + 1)\phi_x^2 - (\beta - 1)\phi_x^2 - \gamma\phi_x\phi_t$, for a logarithmic variable ϕ . We demonstrated that using the modified equation instead of the original one dramatically improves the accuracy and long-term stability of numerical integration.

There is no reason to believe that asymptotic instability is limited to scalar equations and that it cannot occur in more complex systems such as equations of GR. We want to investigate whether discretized Einstein's equations are prone to such an instability, and if numerical integration of these equations can be improved and prolonged by an exponential transformation of variables. As a first step in this direction, we show that it is possible to formulate a tensorial exponential transformation for a three-dimensional metric γ_{ij} of space-like hypersurfaces embedded in a four-dimensional space-time (Section 2). The transformation leads to a 3+1 formulation of Einstein's equations which describes the evolution of γ_{ij} at every point of a hypersurface as stretch and rotation of a local cartesian coordinate system in a tangential space (Section 3).

2 Exponential transformation of a three-dimensional metric

2.1 Metric in terms of stretch and rotation variables

To carry out a tensorial exponential transformation, we write a three-dimensional metric of a space-like hypersurface, $dh^2 = \gamma_{ij}dx^i dx^j$, as

$$\gamma_{ij} = A_{ik}^\dagger D_{kl} A_{lj}, \quad (2.1)$$

where $D_{ij} = \delta_{ij}\lambda_i$ is a diagonal matrix of eigenvalues of γ_{ij} , and A_{ij} is the orthogonal matrix of rotations, $A_{ij}^\dagger = A_{ij}^{-1}$; superscript \dagger denotes a matrix transposition, $A_{ij}^\dagger = A_{ji}$, and $A_{mi}A_{mj} = \delta_{ij}$. Decomposition (2.1) is always possible for a symmetric matrix.

Since the three-dimensional metric is positive definite, $0 < \lambda_i < \infty$, we can introduce logarithms of eigenvalues $\phi_i = \ln \lambda_i$ and write

$$D_{ij} = \delta_{ij}e^{\phi_i}, \quad -\infty < \phi_1, \phi_2, \phi_3 < \infty. \quad (2.2)$$

The transformation maps eigenvalues λ_i onto a $(-\infty, \infty)$ interval; $dh^2 = 0$ corresponds to $\phi_k \rightarrow -\infty$.

Rotation matrix can be further expressed as exponentiation of an anti-symmetric matrix. We write

$$A_{ij} = \exp(M_{ij}), \quad M_{ij} = \epsilon_{ijk}\theta_k, \quad -\infty > \theta_1, \theta_2, \theta_3 > -\infty, \quad (2.3)$$

where θ_k is a rotation vector dual to M_{ij} , and ϵ_{ijk} is a fully anti-symmetric symbol (matrices $w_{ij}^k = \epsilon_{ijk}$, $k = 1, 2, 3$, are generators of a rotation group). Using new variables, we can rewrite (2.1) as

$$\gamma_{ij} = e^{\epsilon_{ikm}\theta_m} e^{\phi_k} e^{-\epsilon_{jkn}\theta_n}. \quad (2.4)$$

Equation (2.1) or (2.4) describes γ_{ij} as a result of stretch and rotation of a cartesian coordinate system with a metric δ_{ij} carried out in a tangential space at every point of a hypersurface. Transformation (2.1) is unique if all three eigenvalues of γ_{ij} are different. If two or three eigenvalues are degenerate, then one of the angles of rotation or all three of them cannot be uniquely defined. However, quantities which have physical meaning, such as angular velocities and acceleration, remain meaningful. The evolution equations for ϕ_i and θ_i which we formulate below are unique (we address this in detail at the end of the paper). In what follows, we will refer to ϕ_i , θ_i , and their certain derivatives as exponential stretch rotation (ESR) variables. Our aim will be to rewrite Einstein's equations in terms of ESR variables.

Before proceeding any further, we must discuss index notation and summation rules used in the paper. These rules are different from these commonly used in GR because ESR variables and matrices A_{ij} do not transform like vectors or tensors under curvilinear coordinate transformations, and it usually takes a combination of several ESR variables to form a covariant or a contravariant object. In index notations, we will not distinguish between upper and lower indices in ESR variables and their derivatives. For example, A_j^i will be a matrix with the same elements as A_{ij} or A^{ij} . Summation in formulas involving ESR variables is implicitly assumed over a repeating index. An index in a formula may repeat more than two times. However, there will be one (and only one) very important exception from this rule: summation must never be carried over a repeating index if the index is present *only once* on one side of an equation and more than once on the other side. Equation (2.2) above is an example of the

summation exception rule. There is no summation over i in this formula. Another example will be an equation $a_j^i = b_{lm}^i c_{kl}^i d_{ijkl}^m$, where summation must be carried out over indices m , k , and l , but no summation is carried over i . However, in the equation $a_i^i = b_{lm}^i c_{kl}^i d_{iikl}^m$ summation must be carried over m , k , l , and i on the right-hand side, and over i on the left-hand side. These modified rules let one to simplify and unambiguously perform manipulations with equations involving ESR variables, and to express tensorial objects in terms of ESR variables in a compact and unambiguous way.

In what follows, it will be convenient to use auxiliary variables

$$\eta_k = \frac{\theta_k}{\theta}, \quad \theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}, \quad -1 < \eta_k < 1. \quad (2.5)$$

Since $\epsilon_{nik}\epsilon_{njm} = \delta_{ij}\delta_{km} - \delta_{im}\delta_{kj}$, we have

$$M_{ij}^2 = \epsilon_{ink}\theta_k\epsilon_{njm}\theta_m = \theta^2(\eta_i\eta_j - \delta_{ij}). \quad (2.6)$$

Rodrigues formula then gives the rotation matrix A_{ij} in terms of θ_k ,

$$\begin{aligned} A_{ij} &= \delta_{ij} + M_{ij} \frac{\sin \theta}{\theta} + M_{ij}^2 \frac{1 - \cos \theta}{\theta^2} \\ &= \delta_{ij} \cos \theta + \epsilon_{ijk}\eta_k \sin \theta + \eta_i\eta_j(1 - \cos \theta). \end{aligned} \quad (2.7)$$

Using (2.7), we can write a closed expression for the metric (2.1) as

$$\begin{aligned} \gamma_{ij} &= \delta_{ij} e^{\phi_i} \cos^2 \theta + (e^{\phi_i} - e^{\phi_j}) \epsilon_{ijk}\eta_k \sin \theta \cos \theta + \\ &\quad \epsilon_{ink}\epsilon_{jnm}\eta_m \eta_k e^{\phi_n} \sin^2 \theta + (e^{\phi_i} + e^{\phi_j}) \eta_i\eta_j \cos \theta(1 - \cos \theta) - \\ &\quad (\epsilon_{ink}\eta_j + \epsilon_{jnk}\eta_i) e^{\phi_n} \eta_k \eta_n \sin \theta(1 - \cos \theta) + e^{\phi_n} \eta_n^2 \eta_i\eta_j(1 - \cos \theta)^2. \end{aligned} \quad (2.8)$$

Since

$$\gamma_{ij} = A_{mi} e^{\phi_m} A_{mj}, \quad \gamma^{ij} = (A_{mi} e^{\phi_m} A_{mj})^{-1} = A_{mi} A_{mj} e^{-\phi_m}, \quad (2.9)$$

formula (2.8) with $\phi_i \rightarrow -\phi_i$ gives a closed expression for γ^{ij} .

2.2 Differentials of the metric

We now must derive the formulas which relate first and second differentials of the metric, $d\gamma_{ij}$ and $d^2\gamma_{ij}$, with first and second differentials of ESR variables, $d\phi_i$, $d\theta_i$, $d^2\phi_i$, $d^2\theta_i$. Differentiation of (2.1) gives

$$\begin{aligned} d\gamma_{ij} &= dA_{in}^\dagger D_{nm} A_{mj} + A_{in}^\dagger D_{nm} dA_{mj} + A_{in}^\dagger dD_{nm} A_{mj} \\ &= \left(B_{k;ni} e^{\phi_n} A_{nj} + A_{ni} e^{\phi_n} B_{k;nj} \right) d\theta_k + \left(A_{ki} e^{\phi_k} A_{kj} \right) d\phi_k, \end{aligned} \quad (2.10)$$

where we introduced three new matrices B_k which are first derivatives of a rotation matrix with respect to θ_k ,

$$\begin{aligned} B_{k;ij} &\equiv \frac{\partial A_{ij}}{\partial \theta_k} = \epsilon_{ijk} \frac{\sin \theta}{\theta} + (\delta_{ik}\eta_j + \delta_{jk}\eta_i) \frac{1 - \cos \theta}{\theta} + \\ &\quad \eta_k \left(-\sin \theta \delta_{ij} + \epsilon_{ijn}\eta_n \left(\cos \theta - \frac{\sin \theta}{\theta} \right) + \eta_i\eta_j \left(\sin \theta - 2 \frac{1 - \cos \theta}{\theta} \right) \right). \end{aligned} \quad (2.11)$$

In (2.11) and throughout the paper, symbol ";" does not mean a covariant differentiation, but simply used to separate indices of different nature.

To invert (2.10) and obtain $d\phi_k$ and $d\theta_k$ in terms of $d\gamma_{ij}$, we must rotate (2.10) into a coordinate system where γ_{ij} is diagonal by multiplying (2.10) with A_{ij} and A_{ij}^\dagger . The result is

$$A_{in}d\gamma_{nm}A_{mj}^\dagger = C_{k;ij}d\theta_k + \delta_{ij}e^{\phi_i}d\phi_i, \quad (2.12)$$

where

$$C_{k;ij} \equiv A_{in}B_{k;nm}^\dagger D_{mj} + D_{in}B_{k;nm}A_{mj}^\dagger = A_{in}B_{k;jn}e^{\phi_j} + A_{jn}B_{k;in}e^{\phi_i}. \quad (2.13)$$

From orthogonality of A_{ij} it follows that combinations $A_{in}B_{k;nj}^\dagger$ are anti-symmetric,

$$A_{in}B_{k;nj}^\dagger = A_{in}B_{k;jn} = -B_{k;in}A_{jn} = -A_{jn}B_{k;ni}^\dagger. \quad (2.14)$$

Therefore, $C_{k;ij}$ are symmetric trace-free matrices with all diagonal elements identically equal to zero,

$$C_{k;ij} = C_{k;ji}, \quad C_{k;11} = C_{k;22} = C_{k;33} = 0, \quad (2.15)$$

and because of this property $d\phi_k$ and $d\theta_k$ in (2.12) do not mix. We can use diagonal part of (2.12) to find

$$d\phi_i = e^{-\phi_i}A_{im}A_{in}d\gamma_{mn}. \quad (2.16)$$

Off-diagonal part of (2.12) gives a system of three linear equations

$$A_{in}d\gamma_{nm}A_{jm} = C_{k;ij}d\theta_k, \quad i \neq j, \quad (2.17)$$

which can be solved to find $d\theta_i$,

$$d\theta_i = C_{ij}^{-1}|\epsilon_{jmn}|A_{mr}A_{nk}d\gamma_{rk}, \quad (2.18)$$

where

$$C_{ij} = |\epsilon_{inm}|C_{j;nm}, \quad (2.19)$$

and $|\cdot|$ is a module operation. Using matrices $C_{k;ij}$ we can write

$$d\gamma_{ij} = A_{ki}A_{kj}e^{\phi_k}d\phi_k + A_{ni}A_{mj}C_{k;nm}d\theta_k. \quad (2.20)$$

To derive expressions for $d^2\phi_i$ and $d^2\theta_i$ we differentiate (2.12) and obtain

$$\begin{aligned} \delta_{ij}e^{\phi_i}d^2\phi_i + C_{k;ij}d^2\theta_k = \\ A_{in}d^2\gamma_{nm}A_{jm} + \mathcal{H}_{ij;nm}^0d\phi_n d\phi_m + \mathcal{H}_{ij;nm}^1d\theta_n d\theta_m + \mathcal{H}_{ij;nm}^2d\phi_n d\theta_m, \end{aligned} \quad (2.21)$$

where

$$\mathcal{H}_{ij;nm}^0 = -\delta_{ij}\delta_{im}\delta_{jn}e^{\phi_i}, \quad (2.22)$$

$$\mathcal{H}_{ij;nm}^1 = H_{ij;nm}^1 + H_{ji;nm}^1, \quad \mathcal{H}_{ij;nm}^2 = H_{ij;nm}^2 + H_{ji;nm}^2, \quad (2.23)$$

$$\mathcal{H}_{ij;nk}^1 = B_{n;im}A_{rm}C_{k;rj} - e^{\phi_i}(B_{k;im}B_{n;jm} + E_{kn;im}A_{jm}), \quad (2.24)$$

$$H_{ij;nk}^2 = B_{k;im} A_{jm} \left(e^{\phi_j} \delta_{jn} - e^{\phi_i} \delta_{in} \right), \quad (2.25)$$

and we denoted second derivatives of a rotation matrix as

$$\begin{aligned} E_{kr;ij} \equiv \frac{\partial^2 A_{ij}}{\partial \theta_k \partial \theta_r} &= (\delta_{ik} \delta_{jr} + \delta_{jk} \delta_{ir}) \frac{1 - \cos \theta}{\theta^2} - \delta_{ij} \delta_{kr} \frac{\sin \theta}{\theta} + \\ &\left(\frac{\cos \theta}{\theta} - \frac{\sin \theta}{\theta^2} \right) (\epsilon_{ijk} \eta_r + \epsilon_{ijr} \eta_k + \epsilon_{ijn} \eta_n (\delta_{kr} - 3\eta_r \eta_k)) - \\ &\epsilon_{ijn} \eta_r \eta_k \eta_n \sin \theta - \delta_{ij} \eta_k \eta_r \left(\cos \theta - \frac{\sin \theta}{\theta} \right) + \\ &(\delta_{ik} \eta_j \theta_r + \delta_{jk} \eta_i \theta_r + \delta_{ir} \eta_j \theta_k + \delta_{jr} \eta_i \theta_k + \delta_{kr} \eta_i \theta_j) \left(\frac{\sin \theta}{\theta} - 2 \frac{1 - \cos \theta}{\theta^2} \right) + \\ &\eta_i \eta_j \eta_k \eta_r \left(\cos \theta - \frac{\sin \theta}{\theta} - 4 \left(\frac{\sin \theta}{\theta} - 2 \frac{1 - \cos \theta}{\theta^2} \right) \right) \end{aligned} \quad (2.26)$$

Note, that $\mathcal{H}_{ij;nm}^0 d\phi_n d\phi_m$ make a non-zero contribution only to a diagonal part of (2.21), whereas $\mathcal{H}_{ij;nm}^2 d\phi_n d\theta_m$ contributes only to an off-diagonal part.

Diagonal part of (2.21) then gives us

$$d^2 \phi_i = e^{-\phi_i} \left(A_{ik} d^2 \gamma_{kn} A_{in} + \mathcal{H}_{ii;nm}^1 d\theta_n d\theta_m \right) - (d\phi_i)^2. \quad (2.27)$$

Off-diagonal part of (2.21) gives

$$d^2 \theta_i = C_{ij}^{-1} |\epsilon_{jmn}| \left(A_{mr} A_{nk} d^2 \gamma_{rk} + \mathcal{H}_{ij;nm}^1 d\theta_n d\theta_m + \mathcal{H}_{ij;nm}^2 d\phi_n d\theta_m \right), \quad (2.28)$$

where C_{ij} has been defined in (2.19). The above formulas, (2.16), (2.18), (2.27), and (2.28), relate first and second differentials of new variables ϕ_k and θ_k to first and second differentials of γ_{ij} . There is a linear relation between $d\theta_k$, $d\phi_k$ and $d\gamma_{ij}$ ((2.12)). Both $d^2 \phi_k$ and $d^2 \theta_k$ depend linearly on second differentials $d^2 \gamma_{ij}$ and in addition involve squares of first differentials $d\gamma_{ij}$ (see (2.21)).

2.3 Cristoffels and Ricci tensor

We use (2.10) to write first derivatives of γ_{ij} with respect to spatial coordinates as

$$\frac{\partial \gamma_{ij}}{\partial x^k} = A_{ai} A_{aj} e^{\phi_a} \frac{\partial \phi_a}{\partial x^k} + e^{\phi_a} (B_{m;ai} A_{aj} + A_{ai} B_{m;aj}) \frac{\partial \theta_m}{\partial x^k}, \quad (2.29)$$

and Cristoffel symbols

$$\Gamma_{ij}^k = \gamma^{kn} \left(\frac{\partial \gamma_{in}}{\partial x^j} + \frac{\partial \gamma_{jn}}{\partial x^i} - \frac{\partial \gamma_{ij}}{\partial x^n} \right) \quad (2.30)$$

as

$$\begin{aligned} \Gamma_{ij}^k = \frac{1}{2} A_{ak} \left(A_{ai} \frac{\partial \phi_a}{\partial x^j} + A_{aj} \frac{\partial \phi_a}{\partial x^i} - e^{\phi_b - \phi_a} A_{am} A_{bi} A_{bj} \frac{\partial \phi_b}{\partial x^m} + \right. \\ \left. \left(B_{m;ai} + e^{\phi_b - \phi_a} A_{an} A_{bi} B_{m;bn} \right) \frac{\partial \theta_m}{\partial x^j} + \right. \\ \left. \left(B_{m;aj} + e^{\phi_b - \phi_a} A_{an} A_{bj} B_{m;bn} \right) \frac{\partial \theta_m}{\partial x^i} - \right. \\ \left. \left(B_{m;bi} A_{bj} + A_{bi} B_{m;bj} \right) e^{\phi_b - \phi_a} A_{an} \frac{\partial \theta_m}{\partial x^n} \right) \end{aligned} \quad (2.31)$$

From (2.31) we get contracted Cristoffel symbols

$$\Gamma_i = \Gamma_{ik}^k = \frac{1}{2} \frac{\partial \ln \det(\gamma_{ij})}{\partial x^i} = \frac{1}{2} \sum_a \frac{\partial \phi_a}{\partial x^i}, \quad (2.32)$$

and their spatial derivatives

$$\frac{\partial \Gamma_i}{\partial x^j} = \frac{1}{2} \sum_a \frac{\partial^2 \phi_a}{\partial x^i \partial x^j}. \quad (2.33)$$

Γ_i do not depend on partial derivatives of angular variables θ_i . Expressions for spatial derivatives of Cristoffel symbols are

$$\begin{aligned} \frac{\partial \Gamma_{ij}^a}{\partial x^a} = & (1) \Gamma_{ij;rkcn} \frac{\partial^2 \phi_r}{\partial x^k \partial x^n} + (2) \Gamma_{ij;rkcn} \frac{\partial^2 \theta_r}{\partial x^k \partial x^n} + \\ & (3) \Gamma_{ij;rskn} \frac{\partial \phi_r}{\partial x^k} \frac{\partial \phi_s}{\partial x^n} + (4) \Gamma_{ij;rskn} \frac{\partial \theta_r}{\partial x^k} \frac{\partial \theta_s}{\partial x^n} + (5) \Gamma_{ij;rskn} \frac{\partial \phi_s}{\partial x^n} \frac{\partial \theta_r}{\partial x^k}, \end{aligned} \quad (2.34)$$

where

$$(1) \Gamma_{ij;rkcn} = \frac{1}{2} \left(A_{rk} A_{ri} \delta_{nj} + A_{rk} A_{rj} \delta_{ni} - e^{\phi_r - \phi_a} A_{an} A_{ak} A_{ri} A_{rj} \right), \quad (2.35)$$

$$(2) \Gamma_{ij;rskn} = \frac{1}{2} A_{sk} A_{sn} A_{ri} A_{rj} e^{\phi_r - \phi_s} - \frac{1}{2} A_{ak} A_{an} A_{ri} A_{rj} \delta_{sr} e^{\phi_r - \phi_a}, \quad (2.36)$$

$$\begin{aligned} (3) \Gamma_{ij;rkcn} = & \frac{1}{2} A_{ak} \left(B_{r;ai} + e^{\phi_b - \phi_a} A_{ac} A_{bi} B_{r;bc} \right) \delta_{jn} \\ & + \frac{1}{2} A_{ak} \left(B_{r;aj} + e^{\phi_b - \phi_a} A_{ac} A_{bj} B_{r;bc} \right) \delta_{in} \\ & - \frac{1}{2} A_{ak} e^{\phi_b - \phi_a} A_{an} \left(B_{r;bi} A_{bj} + A_{bi} B_{r;bj} \right), \end{aligned} \quad (2.37)$$

$$\begin{aligned} (4) \Gamma_{ij;rskn} = & \frac{1}{2} \left(B_{r;ak} B_{s;ai} + A_{ak} E_{sr;ai} + Q_{ai;rska} \right) \delta_{jn} + \\ & \frac{1}{2} \left(B_{r;ak} B_{s;aj} + A_{ak} E_{sr;aj} + Q_{aj;rska} \right) \delta_{in} - \frac{1}{2} (Q_{ij;rskn} + Q_{ji;rskn}), \end{aligned} \quad (2.38)$$

where

$$\begin{aligned} Q_{ij;rskn} = e^{\phi_b - \phi_a} \left(B_{r;ak} A_{an} B_{s;bi} A_{bj} + A_{ak} B_{r;an} B_{s;bi} A_{bj} + \right. \\ \left. A_{ak} A_{an} E_{sr;bi} A_{bj} + A_{ak} A_{an} B_{s;bi} B_{r;bj} \right), \end{aligned} \quad (2.39)$$

and

$$\begin{aligned}
{}^{(5)}\Gamma_{ij;rskn} &= \frac{1}{2} \left(B_{r;sk} A_{si} + A_{sk} B_{r;si} \right) \delta_{jn} + \frac{1}{2} \left(B_{r;sk} A_{sj} + A_{sk} B_{r;s_j} \right) \delta_{in} \\
&\quad - \frac{1}{2} e^{\phi_s - \phi_a} \left(B_{r;ak} A_{an} + A_{ak} B_{r;an} \right) A_{si} A_{sj} \\
&\quad - \frac{1}{2} e^{\phi_s - \phi_a} A_{ak} A_{an} \left(B_{r;si} A_{sj} + A_{si} B_{r;s_j} \right) \\
&\quad + \frac{1}{2} e^{\phi_s - \phi_a} A_{an} A_{ab} B_{r;sb} (A_{si} \delta_{kj} + A_{sj} \delta_{ki}) \\
&\quad - \frac{1}{2} e^{\phi_a - \phi_s} A_{sn} A_{sb} B_{r;ab} (A_{ai} \delta_{kj} + A_{aj} \delta_{ki}) \\
&\quad - \frac{1}{2} A_{ak} A_{an} e^{\phi_s - \phi_a} (B_{r;si} A_{sj} + A_{si} B_{r;s_j}) \\
&\quad + \frac{1}{2} A_{sk} A_{sn} e^{\phi_b - \phi_s} (B_{r;bi} A_{bj} + A_{bi} B_{r;b_j}).
\end{aligned} \tag{2.40}$$

Three-dimensional Ricci tensor

$$R_{ij} = \frac{\partial \Gamma_{ij}^a}{\partial x^a} - \frac{\partial \Gamma_i}{\partial x^j} + \Gamma_{ij}^a \Gamma_a - \Gamma_{ia}^b \Gamma_{jb}^a \tag{2.41}$$

written in terms of new variables will contain second order quasi-linear terms proportional to $\frac{\partial^2 \phi_r}{\partial x^k \partial x^n}$ and $\frac{\partial^2 \theta_r}{\partial x^k \partial x^n}$ plus lower order non-linear terms made of products of first derivatives $\frac{\partial \phi_s}{\partial x^n}$ and $\frac{\partial \theta_r}{\partial x^k}$. We can write it as

$$\begin{aligned}
R_{ij} &= {}^{(1)}R_{ij;rkcn} \frac{\partial^2 \phi_r}{\partial x^k \partial x^n} + {}^{(2)}R_{ij;rkcn} \frac{\partial^2 \theta_r}{\partial x^k \partial x^n} + \\
&\quad {}^{(3)}R_{ij;rskn} \frac{\partial \phi_r}{\partial x^k} \frac{\partial \phi_s}{\partial x^n} + {}^{(4)}R_{ij;rskn} \frac{\partial \theta_r}{\partial x^k} \frac{\partial \theta_s}{\partial x^n} + {}^{(5)}R_{ij;rskn} \frac{\partial \phi_s}{\partial x^n} \frac{\partial \theta_r}{\partial x^k}.
\end{aligned} \tag{2.42}$$

Using formulas (2.32) - (2.40), coefficients in front of second-order terms can be written as

$${}^{(1)}R_{ij;rkcn} = \frac{1}{2} \left(A_{rk} A_{ri} \delta_{nj} + A_{rk} A_{rj} \delta_{ni} - e^{\phi_r - \phi_a} A_{an} A_{ak} A_{ri} A_{rj} \right) - \frac{1}{2} \delta_{ik} \delta_{jn} \delta_{rr}, \tag{2.43}$$

and

$${}^{(2)}F_{ij;rskn} = \frac{1}{2} A_{ri} A_{rj} \left(A_{sk} A_{sn} e^{\phi_r - \phi_s} - \delta_{sr} A_{ak} A_{an} e^{\phi_r - \phi_a} \right). \tag{2.44}$$

Low-order term coefficients ${}^{(3,4,5)}R$ can be trivially found from (2.32) - (2.40) as well.

2.4 Trace and trace-free part of the first differential

ESR representation of the metric γ_{ij} leads to a natural decomposition of its first differential $d\gamma_{ij}$ onto trace and trace-free parts. Trace-free part depends only on $d\theta_i$ whereas trace part depends only on $d\phi_i$.

We denote trace of a matrix in a tangential space as

$$tr(a_{ij}) \equiv \delta_{ij} a_{ij} = a_{ii}, \tag{2.45}$$

and trace of a tensor in curved space as

$$Tr(a_{ij}) \equiv \gamma^{ij} a_{ij}. \tag{2.46}$$

From (2.20) we have

$$\text{tr}(d\gamma_{ij}) = A_{ni}C_{k;nm}A_{mi}d\theta_k + A_{ni}e^{\phi_n}d\phi_nA_{ni} = C_{k;nn}d\theta_k + \delta_{nn}e^{\phi_n}d\phi_n = e^{\phi_n}d\phi_n. \quad (2.47)$$

Therefore,

$$d\gamma_{ij} = d\gamma_{ij}^t + d\gamma_{ij}^{tf}, \quad d\gamma_{ij}^t = A_{ni}e^{\phi_n}d\phi_nA_{nj}, \quad d\gamma_{ij}^{tf} = A_{ni}C_{k;nm}A_{mj}d\theta_k, \quad (2.48)$$

where superscripts t and tf denote trace- and trace-free parts of $d\gamma_{ij}$, respectively. Analogously,

$$\begin{aligned} \text{Tr}(d\gamma_{ij}) &= \gamma^{ij}d\gamma_{ij} = A_{ki}e^{-\phi_k}A_{kj} \left(A_{ni}C_{k;nm}A_{mj}d\theta_k + A_{ni}e^{\phi_n}A_{nj}d\phi_n \right) \\ &= C_{k;nn}e^{-\phi_n} + \sum_n d\phi_n = \sum_n d\phi_n \end{aligned} \quad (2.49)$$

so that in a curved space we have as well

$$d\gamma_{ij} = d\gamma_{ij}^T + d\gamma_{ij}^{TF}, \quad d\gamma_{ij}^T = d\gamma_{ij}^t, \quad d\gamma_{ij}^{TF} = d\gamma_{ij}^{tf}. \quad (2.50)$$

3 Einstein's equations in ESR form

Our starting point is a standard ADM 3+1 formulation which consists of an evolution part

$$\begin{aligned} \frac{\partial\gamma_{ij}}{\partial t} &= -2\alpha K_{ij} + \nabla_i\beta_j + \nabla_j\beta_i \\ \frac{\partial K_{ij}}{\partial t} &= -\nabla_i\nabla_j\alpha + \alpha(R_{ij} + \mathcal{K}_{ij}) + \beta^m\nabla_m K_{ij} + K_{im}\nabla_j\beta^m + K_{jm}\nabla_i\beta^m, \end{aligned} \quad (3.1)$$

and constraints

$$\begin{aligned} C_0 &\equiv R + K^2 - K_{ij}K^{ij} = 0, \\ C_i &\equiv \nabla_j K_i^j - \nabla_i K = 0, \end{aligned} \quad (3.2)$$

where

$$R = \gamma^{ij}R_{ij}, \quad K_i^i = \gamma^{im}K_{jm}, \quad K = K_m^m, \quad (3.3)$$

and we defined

$$\mathcal{K}_{ij} \equiv KK_{ij} - 2K_i^n K_{jn}. \quad (3.4)$$

3.1 Evolution part and constraints

To keep up with the first time – second space derivative form of (3.1), we introduce additional ESR variables, rates of deformation ψ_k and angular velocities ω_k ,

$$\frac{\partial\phi_k}{\partial t} = \psi_k, \quad \frac{\partial\theta_k}{\partial t} = \omega_k. \quad (3.5)$$

Our goal is to rewrite (3.1) in terms of ϕ_k , θ_k , ψ_k and ω_k .

Taking time derivative of the first equation in (3.1) and combining it with the second equation, we get

$$\frac{\partial^2\gamma_{ij}}{\partial t^2} = -2\frac{\partial\alpha}{\partial t}K_{ij} + 2\alpha\nabla_i\nabla_j\alpha - 2\alpha^2(R_{ij} + \mathcal{K}_{ij}) + \mathcal{S}_{ij}, \quad (3.6)$$

where we defined

$$\mathcal{S}_{ij} \equiv \frac{\partial}{\partial t} (\nabla_i \beta_j + \nabla_j \beta_i) - 2\alpha (\beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{jm} \nabla_i \beta^m). \quad (3.7)$$

Next step is to rotate (3.6) into a coordinate system where γ_{ij} is diagonal. Making use of (2.21) we obtain

$$\begin{aligned} \delta_{ij} e^{\phi_i} \frac{\partial \psi_i}{\partial t} + C_{k;ij} \frac{\partial \omega_k}{\partial t} &= -2A_{in} A_{jm} K_{nm} \frac{\partial \alpha}{\partial t} + 2\alpha A_{in} A_{jm} \nabla_n \nabla_m \alpha - 2\alpha^2 A_{in} A_{jm} R_{nm} \\ &\quad - 2\alpha^2 A_{in} A_{jm} \mathcal{K}_{nm} + \mathcal{H}_{ij;nm}^0 \psi_n \psi_m + \mathcal{H}_{ij;nm}^1 \omega_n \omega_m + \mathcal{H}_{ij;nm}^2 \psi_n \theta_m + A_{im} A_{jn} \mathcal{S}_{mn}. \end{aligned} \quad (3.8)$$

First term on the left-hand side of (3.8) has non-zero elements only on the main diagonal whereas the second term has only off-diagonal non-zero elements. We will use this to obtain separate equations for ψ_i and ω_i , but first we need to derive explicit expression for the right-hand side of (3.8) in terms of new variables.

From the first equation in (3.1), making use of (2.10) we obtain

$$K_{ij} = -\frac{1}{2\alpha} A_{ai} e^{\phi_a} A_{aj} \psi_a - \frac{1}{2\alpha} A_{ai} C_{k;ab} A_{bj} \omega_k + \frac{1}{2\alpha} (\nabla_i \beta_j + \nabla_j \beta_i), \quad (3.9)$$

so that

$$K_i^j = \gamma^{jk} K_{ki} = -\frac{1}{2\alpha} A_{ki} A_{kj} \psi_k - \frac{1}{2\alpha} e^{-\phi_a} A_{ai} C_{k;ab} A_{bj} \omega_k + \frac{\gamma^{jk}}{2\alpha} (\nabla_i \beta_k + \nabla_k \beta_i), \quad (3.10)$$

and

$$K = K_i^i = -\frac{1}{2\alpha} \sum_k \psi_k + \frac{\gamma^{ik}}{\alpha} \nabla_i \beta_k = -\frac{\psi}{\alpha} + \frac{\gamma^{ik}}{\alpha} \nabla_i \beta_k \quad (3.11)$$

where

$$\psi = \frac{1}{2} \sum_k \psi_k. \quad (3.12)$$

From (3.9) we see that first term on the right-hand side of (3.8) is simply

$$-2 \frac{\partial \alpha}{\partial t} A_{in} A_{jm} K_{nm} = \frac{\partial \ln \alpha}{\partial t} \left(\delta_{ij} e^{\phi_i} \psi_i + C_{k;ij} \omega_k \right), \quad (3.13)$$

and (3.8) thus can be rewritten as

$$\begin{aligned} \delta_{ij} e^{\phi_i} \left(\frac{\partial \psi_i}{\partial t} - \frac{\partial \ln \alpha}{\partial t} \psi_i \right) + C_{k;ij} \left(\frac{\partial \omega_k}{\partial t} - \frac{\partial \ln \alpha}{\partial t} \omega_k \right) &= \\ 2\alpha A_{in} A_{jm} \nabla_n \nabla_m \alpha - 2\alpha^2 A_{in} A_{jm} R_{nm} + A_{im} A_{jn} \mathcal{S}_{mn} & \\ - 2\alpha^2 A_{in} A_{jm} \mathcal{K}_{nm} + \mathcal{H}_{ij;nm}^0 \psi_n \psi_m + \mathcal{H}_{ij;nm}^1 \omega_n \omega_m + \mathcal{H}_{ij;nm}^2 \psi_n \theta_m. & \end{aligned} \quad (3.14)$$

Using (3.9) - (3.11), we obtain

$$\begin{aligned} \mathcal{K}_{ij} &= \frac{\psi}{8\alpha^2} \left(A_{ai} C_{k;ab} A_{bj} \omega_k + A_{ai} e^{\phi_a} A_{aj} \psi_a \right) - \\ &\quad \frac{1}{2\alpha^2} \left(e^{-\phi_c} C_{k;cb} A_{bi} C_{r;ce} A_{ej} \omega_r \omega_k + e^{\phi_c} A_{ci} A_{cj} \psi_c \psi_c + A_{ci} C_{r;ce} A_{ej} \omega_r \psi_c + C_{k;cb} A_{bi} A_{cj} \psi_c \omega_k \right). \end{aligned} \quad (3.15)$$

Rotation of \mathcal{K}_{ij} gives

$$2\alpha^2 A_{ri} A_{sj} \mathcal{K}_{ij} = \delta_{rs} e^{\phi_r} (\psi \psi_r - \psi_r^2) - C_{n;rs} \omega_n (\psi_r + \psi_s - \psi) - e^{-\phi_c} C_{k;cr} C_{n;cs} \omega_n \omega_k. \quad (3.16)$$

First term in (3.16) has only diagonal, second only off-diagonal, and the third has both diagonal and off-diagonal non-zero elements.

Consider first a diagonal part of (3.14). Gathering diagonal contributions from the last four terms on the right-hand side of (3.14) we obtain a set of evolution equations for deformation rates ψ_i ,

$$\frac{\partial \psi_i}{\partial t} - \frac{\partial \ln \alpha}{\partial t} \psi_i = 2\alpha^2 e^{-\phi_i} A_{in} A_{im} \left(\frac{1}{\alpha} \nabla_n \nabla_m \alpha - R_{nm} \right) - \psi \psi_i + \Psi_{i;kn} \omega_n \omega_k + e^{-\phi_i} A_{im} A_{in} \mathcal{S}_{mn}, \quad (3.17)$$

where

$$\Psi_{i;kn} = \left(e^{\phi_i - \phi_c} - e^{\phi_c - \phi_i} \right) A_{im} A_{ir} B_{k;cm} B_{n;cr}, \quad \sum_i \Psi_{i;kn} = 0. \quad (3.18)$$

Gathering off-diagonal contributions in (3.14) we obtain

$$C_{k;ij} \left(\frac{\partial \omega_k}{\partial t} - \frac{\partial \ln \alpha}{\partial t} \omega_k + \psi \omega_k \right) = 2\alpha A_{in} A_{jm} \nabla_n \nabla_m \alpha - 2\alpha^2 A_{in} A_{jm} R_{nm} + \Omega_{ij;nm}^1 \omega_n \omega_m + \Omega_{ij;nm}^2 \psi_n \omega_m + A_{im} A_{jn} \mathcal{S}_{mn}, \quad (3.19)$$

where

$$\begin{aligned} \Omega_{ij;nm}^1 &= A_{it} A_{js} B_{n;ct} B_{m;cs} (e^{\phi_i + \phi_j - \phi_c} - e^{\phi_c}) \\ &\quad - B_{n;is} B_{m;js} (e^{\phi_j} + e^{\phi_i}) - e^{\phi_i} E_{mn;ik} A_{jk} - e^{\phi_j} E_{mn;jk} A_{ik}, \end{aligned} \quad (3.20)$$

$$\Omega_{ij;nm}^2 = A_{ik} B_{m;jk} e^{\phi_j} (\delta_{in} - \delta_{jn}) + A_{jk} B_{m;ik} e^{\phi_i} (\delta_{jn} - \delta_{in}), \quad (3.21)$$

$$\Omega_{ij;nm}^1 = \Omega_{ji;nm}^1, \quad \Omega_{ij;nm}^2 = \Omega_{ji;nm}^2, \quad \Omega_{ii;nm}^2 = 0. \quad (3.22)$$

Finally, solving equations (3.19) we obtain a set of equations for angular velocities ω_i ,

$$\begin{aligned} \frac{\partial \omega_k}{\partial t} - \frac{\partial \ln \alpha}{\partial t} \omega_k + \psi \omega_k &= \\ C_{kr}^{-1} |\epsilon_{rij}| &\left(2\alpha A_{in} A_{jm} \nabla_n \nabla_m \alpha - 2\alpha^2 A_{in} A_{jm} R_{nm} + \Omega_{ij;nm}^1 \omega_n \omega_m + \Omega_{ij;nm}^2 \psi_n \omega_m + A_{im} A_{jn} \mathcal{S}_{mn} \right). \end{aligned} \quad (3.23)$$

Equations (3.17), (3.23) together with two equations (3.5) constitute an evolution part of an ADM system transformed to ESR variables.

Rewriting constraint equations is straightforward but the resulting formulas are rather complicated and we do not present them here. To derive these formulas, one has to substitute expressions for the Riemann tensor (2.42), extrinsic curvature (3.9), (3.10), (3.11), and Cristoffel symbols (2.31) into (3.2). In rewriting the momentum constraints, one has also to differentiate K_j^i and K with respect to x^k , which in turn requires differentiation of terms containing rotation matrices A_{ij} and their first derivatives. Expressions of second-order derivatives of A_{ij} are given in (2.26).

For the energy constraint, one can also obtain a simple formula expressing C_0 through right-hand sides of the evolution equations for ψ_i . Summing up (3.17) and using (3.11) we obtain

$$\begin{aligned} \frac{\partial\psi}{\partial t} - \frac{\partial\ln\alpha}{\partial t}\psi = & -\alpha^2 \left(R + K^2 - \frac{1}{\alpha}\gamma^{nm}\nabla_n\nabla_m\alpha \right) + \\ & 2\psi\gamma^{nm}\nabla_n\beta_m - (\gamma^{nm}\nabla_n\beta_m)^2 + \frac{1}{2}\gamma^{nm}\mathcal{S}_{nm}. \end{aligned} \quad (3.24)$$

First and second terms in brackets in (3.24) can be replaced using the expression (3.2) for the energy constraint,

$$\begin{aligned} \frac{\partial\psi}{\partial t} - \frac{\partial\ln\alpha}{\partial t}\psi = & -\alpha^2 \left(C_0 + K_{nm}K^{nm} - \frac{1}{\alpha}\gamma^{nm}\nabla_n\nabla_m\alpha \right) + \\ & 2\psi\gamma^{nm}\nabla_n\beta_m - (\gamma^{nm}\nabla_n\beta_m)^2 + \frac{1}{2}\gamma^{nm}\mathcal{S}_{nm}. \end{aligned} \quad (3.25)$$

The latter equation can be inverted to obtain the energy constraint as

$$\begin{aligned} \alpha^2 C_0 = & -\frac{1}{2} \sum \frac{\partial\psi_k}{\partial t} + \frac{\partial\ln\alpha}{\partial t}\psi - \alpha^2 \left(K_{nm}K^{nm} - \frac{1}{\alpha}\gamma^{nm}\nabla_n\nabla_m\alpha \right) + \\ & 2\psi\gamma^{nm}\nabla_n\beta_m - (\gamma^{nm}\nabla_n\beta_m)^2 + \frac{1}{2}\gamma^{nm}\mathcal{S}_{nm}. \end{aligned} \quad (3.26)$$

We see that in ESR formulation the energy constraint depends linearly on time derivatives of ψ_i . Setting $C_0 = 0$ in (3.26) will define a plane in the three-dimensional space of $\frac{\partial\psi_i}{\partial t}$ in which the evolution of a constrained solution must occur, regardless of a choice of gauge. The unit normal to the plane is a vector with components $\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}$. A distance from the plane is given by the actual value of $-\alpha^2 C_0$. Modifying (3.17) by adding the term $-\alpha^2 C_0/\sqrt{3}$ to each of the right-hand sides will project numerical evolution onto a local $C_0 = 0$ plane, and may help with improving the accuracy of the evolution along a (generally curved) surface of constrained solutions.

We also note that differentiation of (3.11) gives

$$\frac{\partial K}{\partial t} = -\frac{1}{\alpha} \left(\frac{\partial\psi}{\partial t} - \psi \frac{\partial\ln\alpha}{\partial t} \right) + \frac{\partial}{\partial t} \left(\frac{\gamma^{nm}\nabla_n\beta_m}{\alpha} \right), \quad (3.27)$$

so that we can rewrite (3.24), assuming $C_0 = 0$, in a more familiar form as

$$\frac{\partial K}{\partial t} = \alpha K_{nm}K^{nm} - \gamma^{nm}\nabla_n\nabla_m\alpha + \text{"shift - dependent terms"}. \quad (3.28)$$

3.2 Case of degenerate eigenvalues

As was mentioned in the beginning of the paper, decomposition (2.1) is always possible but it is not unique if γ_{ij} has degenerate eigenvalues. In case of two degenerate eigenvalues, rotation angle in the direction orthogonal to the corresponding eigenvectors is arbitrary. In case of three degenerate eigenvalues all three rotation angles are arbitrary. For example, a flat three-dimensional hypersurface with Cartesian coordinate system on it is a case of triply-degenerate eigenvalues. In all of these cases, the matrix C_{ij} in (2.19) becomes degenerate and cannot be

inverted. Equation (3.23) then cannot be used to determine $\frac{\partial\omega_i}{\partial t}$. Below we describe how $\frac{\partial\omega_i}{\partial t}$ should be determined when eigenvalues are degenerate.

Consider first the case of triple degeneracy, $\phi_1 = \phi_2 = \phi_3$. Using (2.14) we can rewrite (2.13) as

$$C_{k;ij} = A_{in}B_{k;jn}(e^{\phi_j} - e^{\phi_i}). \quad (3.29)$$

We see that in a triply-degenerate case all three $C_{k;ij} \equiv 0$, and (3.19) becomes a system of three algebraic equations

$$2\alpha A_{in}A_{jm}\nabla_n\nabla_m\alpha - 2\alpha^2 A_{in}A_{jm}R_{nm} + \Omega_{ij;nm}^1\omega_n\omega_m + \Omega_{ij;nm}^2\psi_n\omega_m + A_{im}A_{jn}\mathcal{S}_{mn} = 0 \quad (3.30)$$

with respect to ω_i . We assume that conditions for lapse α and shift β_i are given, and that ϕ_i , θ_i , ψ_i , and ω_i are known on a three-dimensional hypersurface as a result of previous evolution. Thus, first, second, and last terms in (3.30) are known, and $\Omega_{ij;nm}^1$ and $\Omega_{ij;nm}^2$ which depend on ϕ_i and θ_i are known as well. Then (3.30) is a set of three quadratic equations for ω_i which are automatically satisfied at a degenerate point by continuity of ω_i . Differentiation of (3.30) with respect to t will give us expressions for $\frac{\partial\omega_i}{\partial t}$ which will depend on ϕ_i , θ_i , ψ_i , ω_i and on $\frac{\partial\psi_i}{\partial t}$. The latter values are given by (3.17). Thus we can uniquely determine $\frac{\partial\omega_i}{\partial t}$ as functions of ϕ_i , θ_i , ψ_i , ω_i , and their spatial derivatives, which was our original purpose.

The case of two degenerate eigenvalues can be treated similarly. Without loss of generality, assume that $\phi_1 = \phi_2$. Then all elements of symmetric matrices $C_{k;ij}$ are zero excluding $C_{k;13} = C_{k;31} \neq 0$ and $C_{k;23} = C_{k;32} \neq 0$. Equations (3.19) then reduce to one algebraic equation with respect to ω_i ,

$$2\alpha A_{1n}A_{2m}\nabla_n\nabla_m\alpha - 2\alpha^2 A_{1n}A_{2m}R_{nm} + \Omega_{12;nm}^1\omega_n\omega_m + \Omega_{12;nm}^2\psi_n\omega_m + A_{1n}A_{2m}\mathcal{S}_{nm} = 0, \quad (3.31)$$

plus two partial differential equations

$$C_{k;13} \left(\frac{\partial\omega_k}{\partial t} - \frac{\partial\ln\alpha}{\partial t}\omega_k + \psi\omega_k \right) = 2\alpha A_{1n}A_{3m}\nabla_n\nabla_m\alpha - 2\alpha^2 A_{1n}A_{3m}R_{nm} + \Omega_{13;nm}^1\omega_n\omega_m + \Omega_{13;nm}^2\psi_n\omega_m + A_{1n}A_{3m}\mathcal{S}_{nm}, \quad (3.32)$$

$$C_{k;23} \left(\frac{\partial\omega_k}{\partial t} - \frac{\partial\ln\alpha}{\partial t}\omega_k + \psi\omega_k \right) = 2\alpha A_{2n}A_{3m}\nabla_n\nabla_m\alpha - 2\alpha^2 A_{2n}A_{3m}R_{nm} + \Omega_{23;nm}^1\omega_n\omega_m + \Omega_{23;nm}^2\psi_n\omega_m + A_{2n}A_{3m}\mathcal{S}_{nm}. \quad (3.33)$$

Differentiation of (3.31) allows us to determine one time derivative say, $\frac{\partial\omega_3}{\partial t}$ as a function of other variables and two other derivatives, $\frac{\partial\omega_1}{\partial t}$, $\frac{\partial\omega_2}{\partial t}$. Substitution of $\frac{\partial\omega_3}{\partial t}$ into (3.32) and (3.33) will then give us two equations to explicitly determine $\frac{\partial\omega_1}{\partial t}$, $\frac{\partial\omega_2}{\partial t}$. Using them we can find $\frac{\partial\omega_3}{\partial t}$.

A similar approach should be used in case of degenerate eigenvalues to determine SRE variables and their derivatives from γ_{ij} and its derivatives on the initial hypersurface. We see from the above consideration that starting from degenerate initial conditions with arbitrary rotation angles θ_i does not mean that rotation velocities ω_i and accelerations $\frac{\partial\omega_i}{\partial t}$ can also be set arbitrary. Angular velocities must be determined from (3.30), and they will depend among other things on lapse and shift conditions that we wish to use for the evolution.

4 Conclusions

In this paper we formulated a tensorial exponential transformation of a three-dimensional metric of space-like hypersurfaces embedded in a four-dimensional space time in terms of exponential stretch-rotation, or ESR variables (Section 2). We derived formulas relating derivatives of the metric with the corresponding derivatives of these variables. A 3+1 system of Einstein's equations, formulated in terms of ESR variables, describes time evolution of the metric at every point of a hypersurface as a continuous stretch and rotation of a local cartesian coordinate system in a tangential space (Section 3).

We want to mention that certain exponential transformation of variables were used in GR before for special symmetric cases. Such transformations sometime simplify analytical operations, and final results may sometime be formulated in a more compact form [6]. The transformation (2.1), (2.4), (2.8) considered in this paper is the most general exponential transformation which can be carried out for a three-dimensional metric without assuming any symmetries.

A potential advantage of the ESR formulation is a change it introduces to the structure of right-hand sides of Einstein's equations by eliminating contravariant four-dimensional metric g^{ab} as a multiplicative term in these equations. For a scalar non-linear hyperbolic equation considered in [5], the removal of a term g^{-1} , where g is a scalar equivalent of the metric, lead to a dramatic improvement in long-term stability of numerical integration. Consider right-hand sides of (3.17) and (3.23). Ricci tensor R_{ij} and other terms written using ESR variables consist of parts that either do not have multipliers $\propto e^{-\phi_i}$ or have multipliers of the type $\propto e^{\phi_i - \phi_j}$. That is, they either do not have γ^{ij} multipliers or have multiplier that are ratios of different eigenvalues of γ_{ij} . In front of the right-hand sides There are multipliers of the type $\alpha^2 e^{-\phi_i}$ in front of the right-hand sides. We must recall that the lapse is a g^{00} component of the four-dimensional metric g_{ab} and thus having this multiplier is equivalent of having ratios of the zero-th (negative) eigenvalue to other three (positive) eigenvalues of a four-dimensional metric. We see that the net result of transformation is replacement of g^{ab} with ratios of four-metric eigenvalues. We hope that the right-hand sides of the evolution equations (3.17) and (3.23) may behave better than the right-hand sides of the original ADM equations.

We want to discuss now differences between the ESR formulation of Einstein's equations and other versions of 3+1 formulations of GR. Zelmanov [7], in his version of a 3+1 decomposition, chooses a congruence of time-like lines (motion of local reference frames), and then at each event on a time-line draws a small element of a three-dimensional space orthogonal to that line (a local three-dimensional reference frame). Einstein's equations in his formulation describe physical accelerations, deformations, and rotation of these reference frames (and physics of matter if space is not empty). There are no global three-dimensional slices embedded in a four-dimensional space-time in his formulation. In a classical ADM 3+1 formulation [1] of GR and subsequent 3+1 formulations, a four-dimensional space-time is split into a family of three-dimensional space-like slices and a universal "time" is introduced to label these slices. A global coordinate system exists on each slice, and a three-dimensional metric of slices evolves with time according to Einstein's equations and pre-determined gauge conditions. In tetrad formulations of GR [8], a family of orthogonal 4-vectors is introduced and Einstein's equations are formulated in terms of variables projected onto tetrad components.

The ESR formulation presented here is closest to a standard 3+1 ADM formulation. How-

ever, instead of metric and its derivatives, it uses variables which can be thought of in the following way. A three-dimensional metric defines an orthogonal *triad* of space-like vectors directed along the main axis of the metric tensor at each point of a hypersurface. ESR variables ϕ_i and θ_i tell how a three-dimensional coordinate system on a hypersurface must be rotated and re-scaled at each point so that it will locally coincide with a three-dimensional Cartesian coordinate system defined by the orthogonal triad at this point. The evolution of ESR variables is equivalent to the evolution of the three-dimensional metric and is governed by Einstein's equations written in terms of these variables plus the pre-defined gauge conditions. There is no orthogonal tetrad in this formulation.

From the above discussion it is clear that ESR 3+1 formulation presented here can be modified and extended by introducing new variables such as spatial derivatives of the metric, and by addition of various combination of constraints, similar to modifications introduced to a standard ADM formulation before [2]. Hyperbolicity and stability properties of ESR system and its modifications will be studied in subsequent publications. We plan to use this new system for numerical solutions of certain problems of GR.

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