Black Hole Quasi-Inspiral:
An Update

Chris Beetle

Florida Atlantic University

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Work with Richard Price, Ben Bromley, John Friedman,
Lior Burko, Koji Uryū, Masaru Shibata, Antonio Tsokaros,
Rob Owen, etc., etc., etc.

The Problem

Gravitational wave observatories (e.g., LIGO) require accurate calculations of waveforms arising from specific astrophysical events.

- Full-scale numerical simulations of the non-linear Einstein equations will likely be possible in the long-term.
- Results will probably (rather, hopefully) be needed sooner.
- In the near-term, approximation techniques are needed.

The quasi-inspiral approach aims to fill a gap in the array of approximation schemes used to model the various stages of black hole inspiral.

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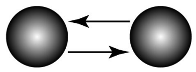

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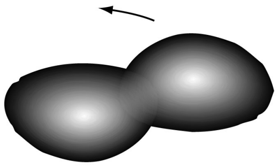
4. Settle-Down



Four Stages of Binary Inspiral

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1. Capture (post-Minkowskian)

   $R_{\text{orbit}} \gg R_{\text{objects}} \qquad T_{\text{decay}} \gg T_{\text{orbit}}$

2. Intermediate (??)

   $R_{\text{orbit}} \sim R_{\text{objects}}, \qquad T_{\text{decay}} \gg T_{\text{orbit}}$

3. Plunge (non-linear evolution)

   $R_{\text{orbit}} \sim R_{\text{objects}}, \qquad T_{\text{decay}} \sim T_{\text{orbit}}$

4. Settle-Down (close limit)

Approximations exist to model all stages but intermediate.

Quasi-Inspiral Approximation

Idea: Since $T_{\text{decay}} \gg T_{\text{orbit}}$ in the intermediate stage, approximate the evolution using a space-time in which no orbital decay occurs.

Stationary Quasi-Inspiral Solve the *exact* field equations, under appropriate boundary conditions, for an *exactly* periodic approximating solution.

+ The approximating space-time will admit a helical Killing vector.

+ The symmetry-reduced equations describe an initial-value problem.

– A helical Killing vector yields a mixed-type initial-value problem. (The domain contains disjoint elliptic and hyperbolic regions.)

– "Appropriate" boundary conditions are subtle in GR.

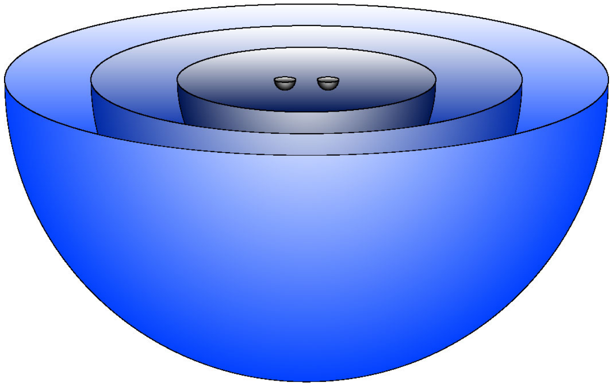
++ One can gain insight into these subtleties by considering toy models. (Quasi-inspiral solutions exist in a wide variety of field theories.)

Effective Linearity

In certain non-linear field theories, although solutions cannot be superposed, it may be possible to *average* two solutions to *approximate* a third.

Hypothesis: GR has this property, called *effective linearity*.

In general, one expects three regions:



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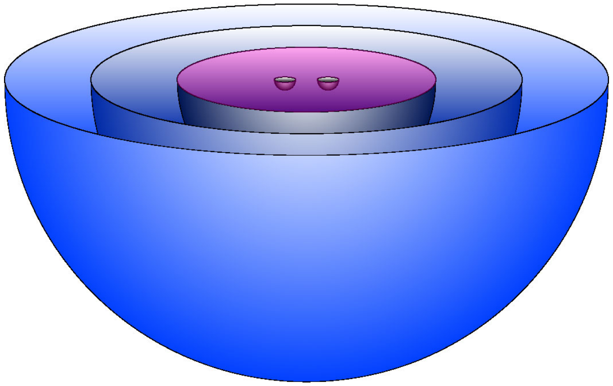

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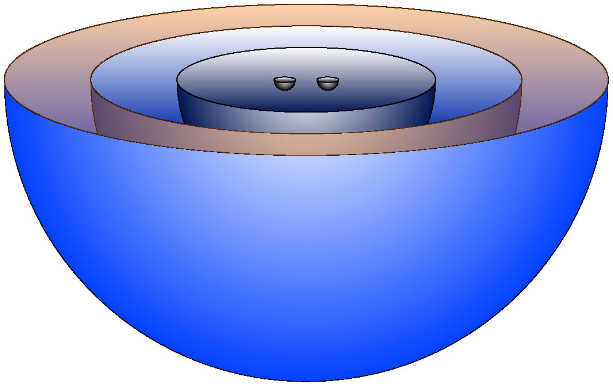

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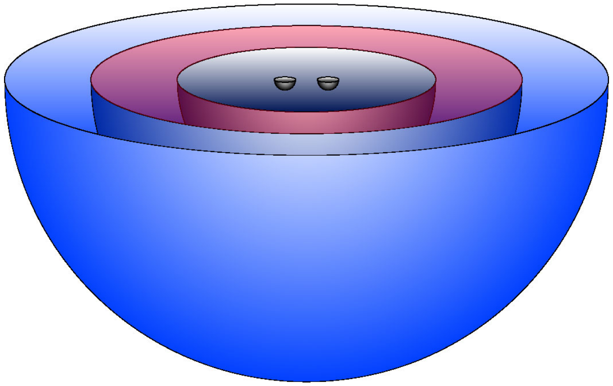
Transition non-linear effects and outer BC's are both relevant.



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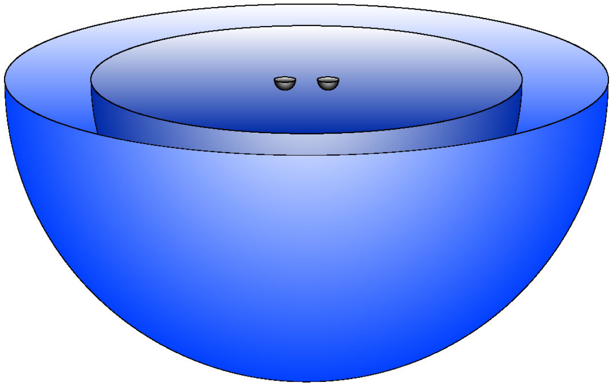
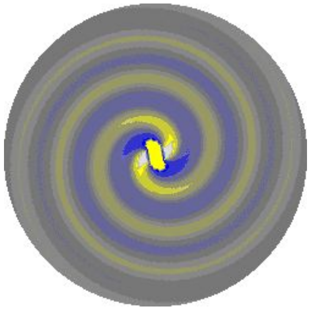


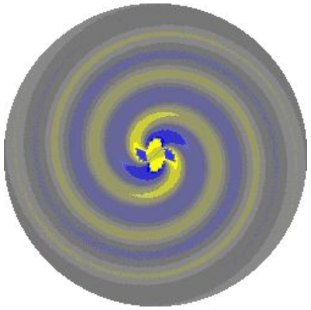
Effective linearity
For sufficiently small Ω (apparently $v \lesssim 0.3$),
the transition region is small enough to ignore.

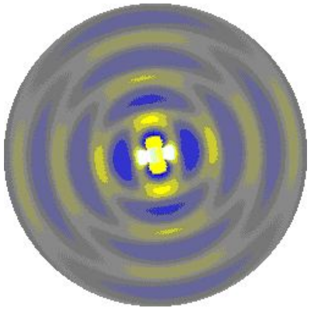
“Standing Wave” Solutions

In many field theories, quasi-inspiral solutions exist describing pure outgoing or pure ingoing radiation.

“Hand of God” Work is done continuously by an external force acting on the sources to maintain constant energy in the (stationary) field.


ingoing radiation


outgoing radiation


radiation balanced

- In linear field theory, radiation balanced solutions need no “hand of god.”

! In GR, *only* radiation balanced solutions can exist.

⇒ Define *periodic standing wave* outer BC’s to pick these solutions.

Applications

If GR exhibits effective linearity, the periodic standing wave solution will approximate the (non-stationary, outgoing) physical solution *everywhere*.

$$\underbrace{\frac{\text{[outgoing]} + \text{[ingoing]}}{2}}_{\sim\text{[PSW]}} + \underbrace{\frac{\text{[outgoing]} - \text{[ingoing]}}{2}}_{\text{small}} = \text{[outgoing]}$$

Intermediate Waveforms The remaining outgoing field in the weak field region will model the outgoing radiation of the physical solution.

Quasi-Stationary Sequences The outgoing flux, and appropriate inner BC's (isolated horizons?), will permit crude models of intermediate dynamics.

Radiation Reaction The linearized ([out] − [in]) solution in the *strong field* region may allow quasi-local calculations of radiation reaction. (Thorne)

Initial Data As quasi-inspiral becomes unstable, one can extract initial data with only outgoing radiation, for full numerical evolution into plunge.

Progress!

“Classical” (Whelan, Krivan, Romano, Price; 98–99)

- 2-d non-linear scalar field theory, numerical solutions

Recent (Price, Friedman, collaborators; now-ish)

- 3-d non-linear scalar field theory with effective linearity; finite differencing, and adapted spectral numerical approaches
- techniques for isolating periodic standing wave solutions (Milwaukee/Utah)
- rigorous existence and uniqueness theorems for 2-d linear theory (Torre)
- coordinate conditions for inspiral space-times in GR

In Progress (1 month – 3 years)

- presentations of adapted spectral techniques and space-time coordinate-fixing scheme
- linearized gravity (build tensor infrastructure)
- Everest (GR)

Adapted Harmonic Coordinates I

Co-Rotating Coordinates

1. Choose a partial Cauchy surface $\Sigma \subset M$, transverse to ξ, and a coordinate patch (x, y, z) thereon.

2. Given Ω, propagate the coordinates to a chart on M using

$$\nabla_\xi x = -\Omega y \qquad \nabla_\xi y = \Omega x \qquad \nabla_\xi z = 0.$$

3. Define a fourth coordinate t via

$$t = 0 \quad \text{on } \Sigma\text{, and} \quad \nabla_\xi t = 1.$$

4. In these (space-time) coordinates, one has the desired result

$$\xi = \frac{\partial}{\partial t} + \Omega\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Adapted Harmonic Coordinates II

Harmonic Co-Rotating Coordinates

1. Decompose the Killing field into its normal and tangent components:
$$\xi^a = N\widehat{n}^a + N^a.$$
2. The wave operator acting on a function f at Σ takes the form
$$\begin{aligned}\nabla^2 f \;\;=\;\; & \left(h^{ab} - N^{-2}N^aN^b\right) D_aD_bf - N^{-2}\left(\nabla_\xi\nabla_\xi f - 2N^aD_a\nabla_\xi f\right) + \\ & N^{-2}\left(ND^bN - N^aD_aN^b\right) D_bf - N^{-1}\left(K + N^{-2}N^aD_aN\right)\nabla_{\widehat{n}} f.\end{aligned}$$
3. Apply this operator to the adapted coordinate functions.
   - $f = t$ gives a restriction on the surface Σ.
   - $f = x, y, z$ gives a set of coupled PDE's.

   \+ If $\nabla^2 f$ vanishes at Σ, it will vanish everywhere on M.

   \+ In specific cases, the equations on Σ admit unique solutions.

Adapted Harmonic Coordinates III

Boundary Conditions

1. Outer boundary conditions on the coordinates are given by matching with rest-frame inertial coordinates of a flat, background metric.
2. Inner boundary conditions are not needed!
    - When the inner boundary of M is null, the principal symbol

      $$q^{ab} = h^{ab} - N^{-2} N^a N^b$$

      is degenerate in the normal direction to $\partial\Sigma$.
    - Analogy: Bessel's equation of order 0

      $$x\, j_0'' + j_0' + x\, j_0 = 0$$

      loses its principal part at $x = 0$. The induced "boundary condition," $j_0' = 0$, selects the *regular* solution $j_0 = J_0$.

   $\Rightarrow$ Regularity conditions pick coordinates uniquely.