Spherical excision for moving black holes and summation by parts for axisymmetric systems

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Outline

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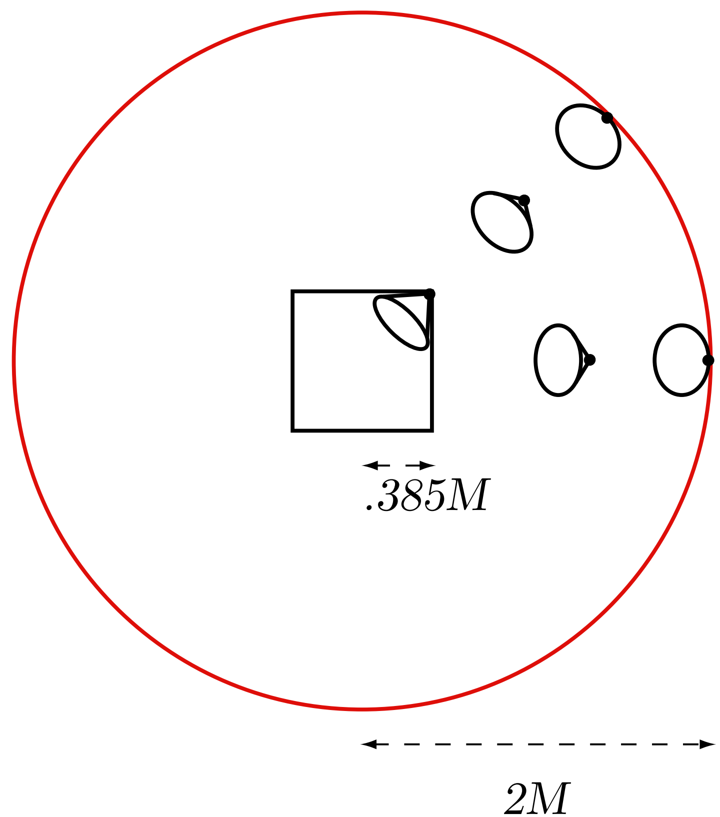
  - Overlapping grids
- Axisymmetric scalar field on a black hole background
  - Single grid energy preserving discretization
  - Spherical excision of a boosted black hole
- Numerical experiments
- Conclusions

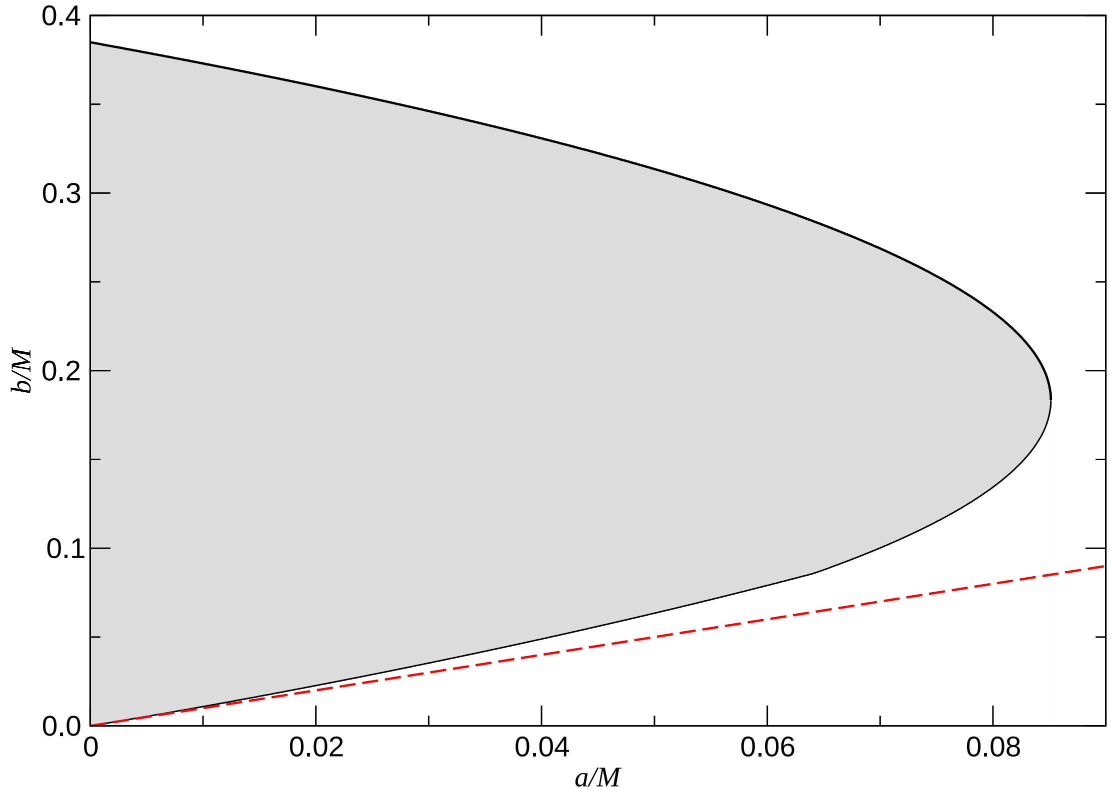
Motivation: Cubical Excision

- Excision: an outflow inner boundary eliminates the singularity.
- For a Schwarzschild BH in KS coordinates the side length of the cube must be smaller than $4\sqrt{3}M/9$.



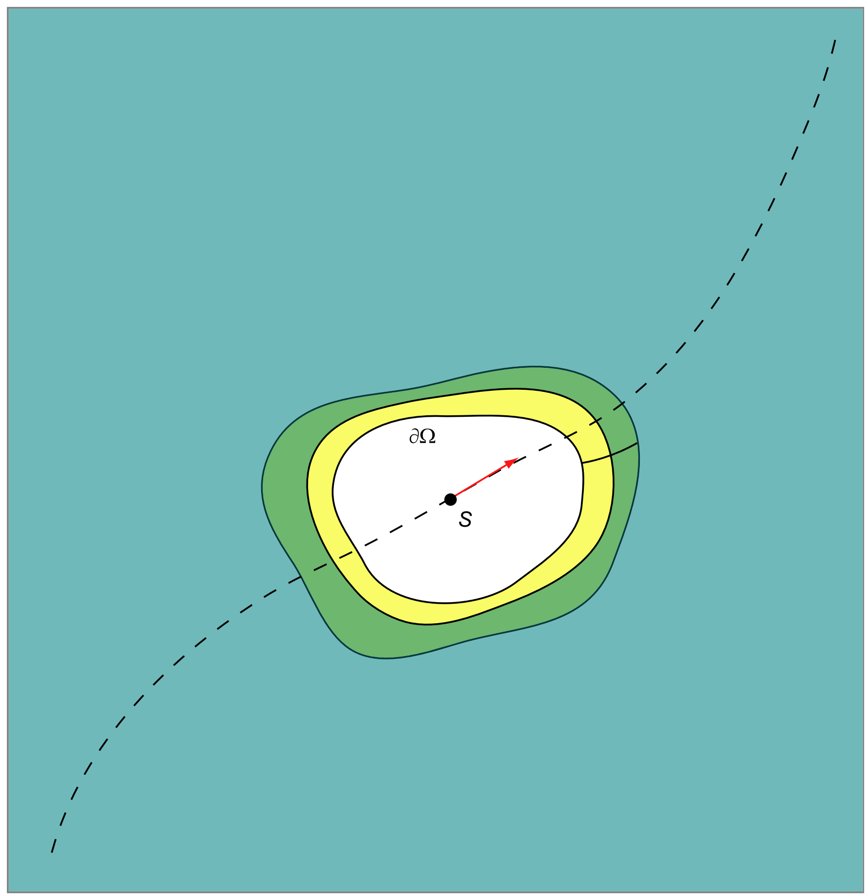
Motivation: Cubical Excision

- For a rotating BH in KS coordinates cubical excision is not possible if $a \gtrsim 0.0851M$.



Multiple Patches

- Hyperbolic initial-boundary value problem in a domain Ω with a moving hole.



Single vs. Overlapping Grids

- Single Cartesian grid
  - Pros
    - Stability proofs
    - Easier to parallelize
  - Cons
    - Cubical domains, complicated algorithms, uniform grid
- Overlapping grids
  - Pros
    - Smooth, time dep. boundaries
    - Well-posedness proofs for hyperbolic problems in general domains
  - Cons
    - Interpolation
    - Few proofs of stability

Massless Klein-Gordon Scalar Field

- We write $\sqrt{-g}\nabla_\mu \nabla^\mu \Phi = \partial_\mu(\gamma^{\mu\nu}\partial_\nu \Phi) = 0$ in first order form

$$
\begin{aligned}
\partial_t \Phi &= T\,, \\
\partial_t T &= -\left(\gamma^{ti}\partial_i T + \partial_i(\gamma^{it}T) + \partial_i(\gamma^{ij}d_j) + \partial_t \gamma^{tt} T + \partial_t \gamma^{tj} d_j\right)/\gamma^{tt}, \\
\partial_t d_i &= \partial_i T\,.
\end{aligned}
$$

- The constraint variables $C_i \equiv d_i - \partial_i \Phi$ propagate trivially.
- Axisymmetry: ∂_ϕ is a spacelike Killing field $\quad\rightarrow\quad \partial_\phi \gamma^{\mu\nu} = 0$.
- Eliminate Φ, d_ϕ: 3×3 system. Introduce $u^T = (T, d_1, d_2)$ and write $\partial_t u = A^i(t,\vec{x})\partial_i u + B(t,\vec{x})u$.

The Energy Method

- In coordinates adapted to a timelike Killing field ∂_t, the energy

$$E = \int_\Omega u^T H u \, d^2x = \int_\Omega (-\gamma^{tt} T^2 + \gamma^{ij} d_i d_j) \, d^2x$$

  is conserved: $\partial_t E = \int_{\partial\Omega} (w_{\text{in}}^2 - w_{\text{out}}^2) \, d\sigma$.

- In KS coordinates the integrand is positive definite where ∂_t is timelike.

- The energy method. Symmetrizable hyperbolic systems with maximal dissipative boundary conditions

$$w_{\text{in}}(t, \vec{x}) = S w_{\text{out}}(t, \vec{x}) + g(t, \vec{x}), \quad \vec{x} \in \partial\Omega$$

  Gives *sufficient* conditions for well posedness.

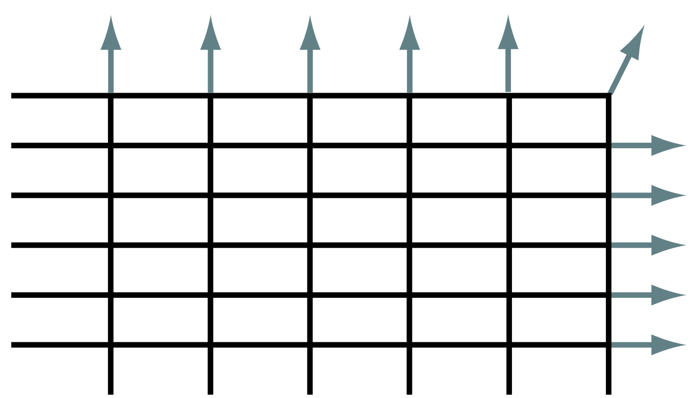
Discretization

- Energy conserving discretization ($E = h_1 h_2 \sum_{ij} u_{ij}^T H_{ij} u_{ij} \sigma_{ij}$):

$$\begin{aligned} \partial_t T &= -\left(\gamma^{ti} D_i T + D_i(\gamma^{it} T) + D_i(\gamma^{ij} d_j) + \partial_t \gamma^{tt} T + \partial_t \gamma^{tj} d_j\right) / \gamma^{tt}, \\ \partial_t d_i &= D_i T\,. \end{aligned}$$

where $(u, Dv)_h = -(Du, v)_h + u_j v_j|_0^N$.

- Boundary data at corners



Axis of Symmetry: Simple Example

- Consider 2D wave equation in polar coordinates

$$\partial_t T = \frac{1}{\rho}\partial_\rho(\rho P)\,; \qquad \partial_t P = \partial_\rho T$$

- Use regularity conditions on the axis. The semidiscrete system

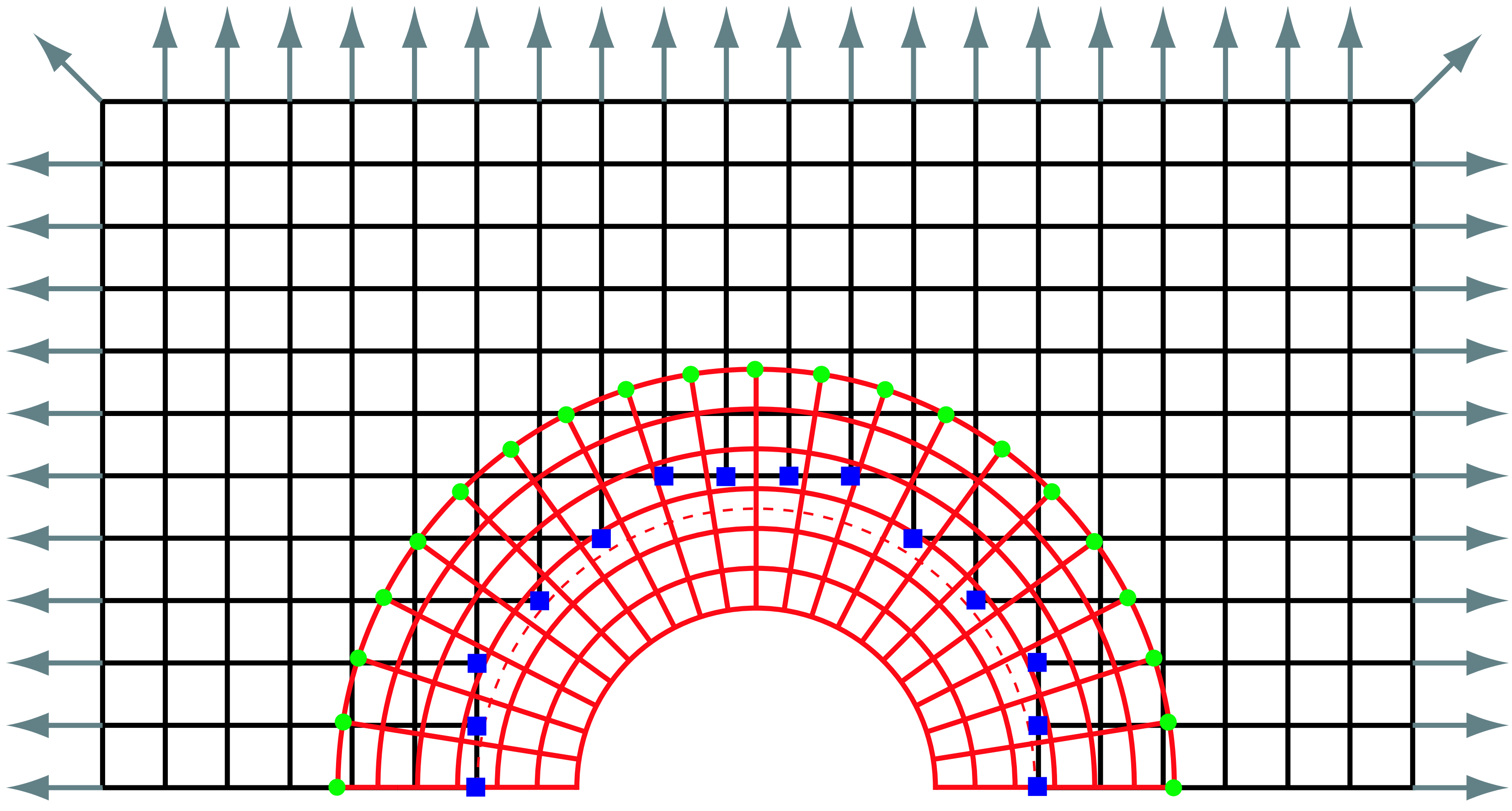
$$\begin{aligned} \partial_t T_i &= \begin{cases} 2D_+ P_0 & i = 0 \\ \frac{1}{\rho_i} D_0(\rho P)_i, & i \geq 1 \end{cases} \\ \partial_t P_i &= D_0 T_i, \quad i \geq 1 \end{aligned}$$

conserves the discrete energy

$$E = \sum_{i=1}^{+\infty} \left(T_i^2 + P_i^2\right)\rho_i \Delta\rho + \frac{1}{4} T_0^2 \Delta\rho^2$$

Spherical Excision

- Introduce a spherical grid adapted to the event horizon



Boosted Black Hole

- Boosted cylindrical KS coordinates $\{\bar{t}, \bar{\rho}, \bar{z}\}$ on the (base) cylindrical grid

$$\bar{t} = \gamma(t - \beta z) \qquad \bar{z} = \gamma(z - \beta t)$$

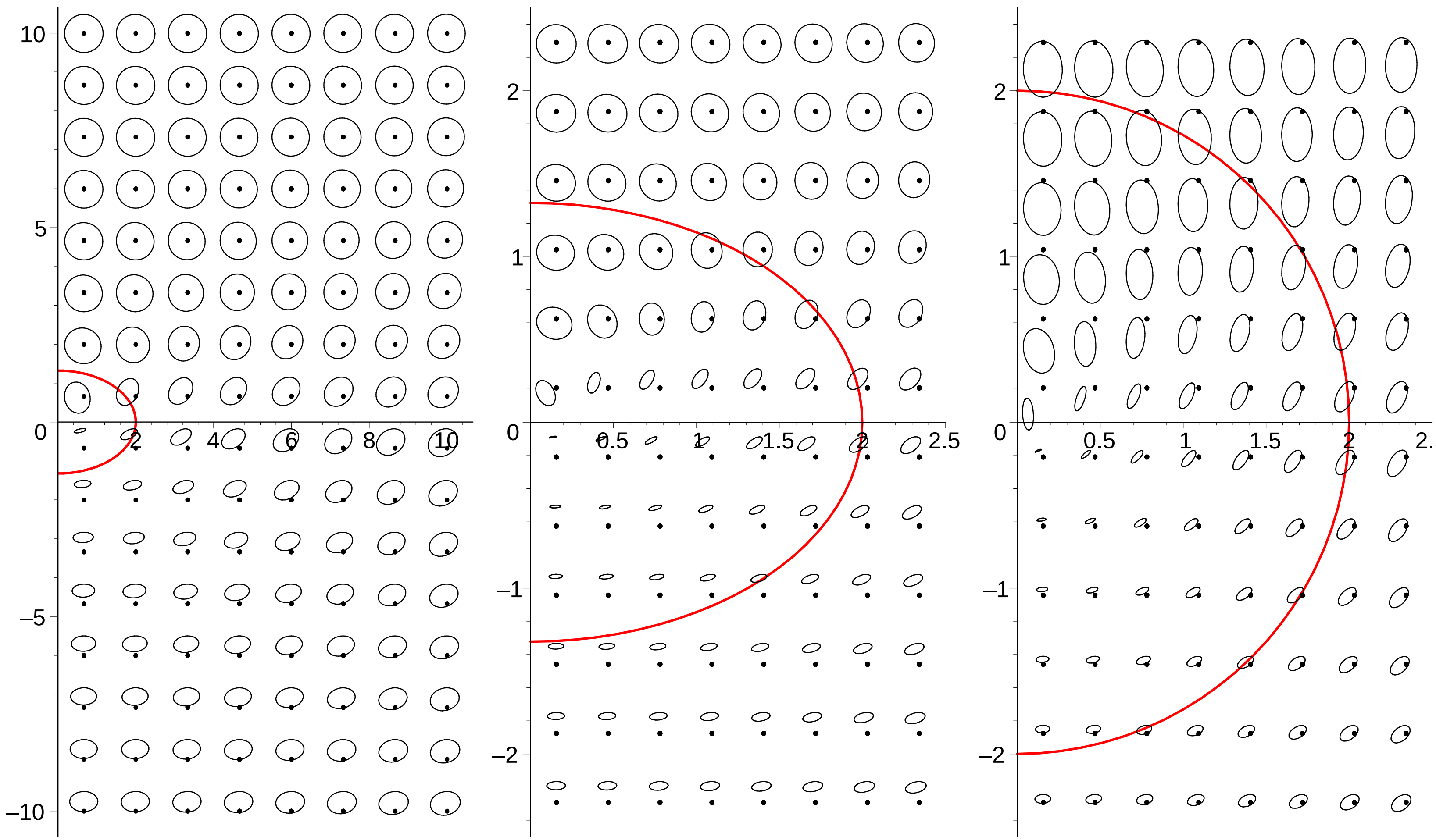
- Co-moving spherical coordinates $\{t', r', \theta'\}$ on the spherical grid

$$t' = \gamma(t - \beta z)$$

- Communication done via interpolation of *all* fields, followed by the transformation law for 1-forms:

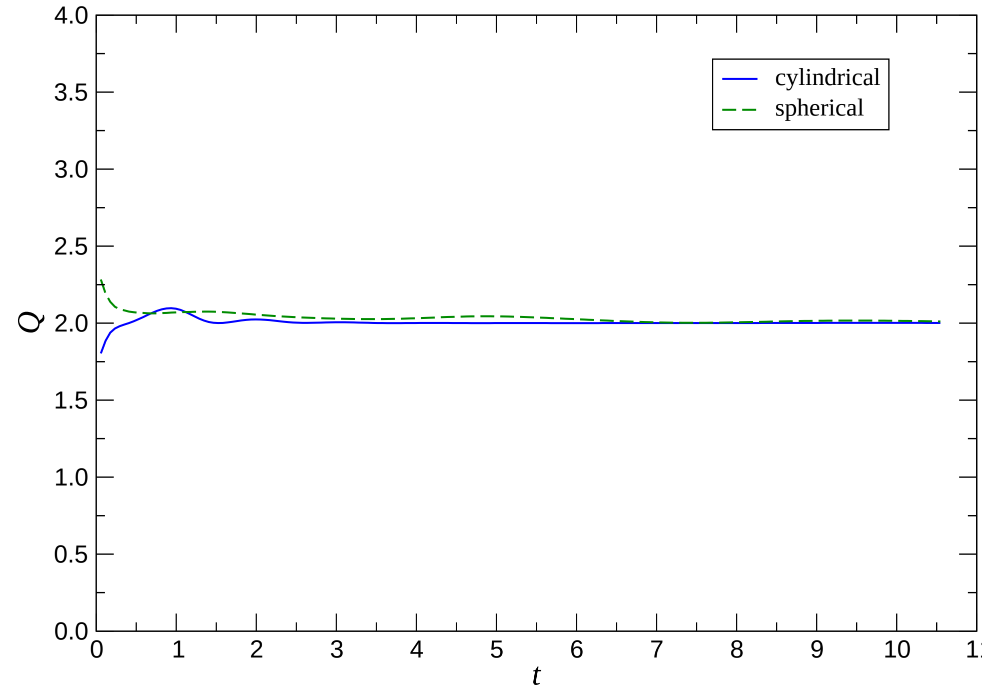
$$\frac{\partial \Phi}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu} \frac{\partial \Phi}{\partial x^\nu}$$

Boosted Black Hole ($\beta = -0.75$)



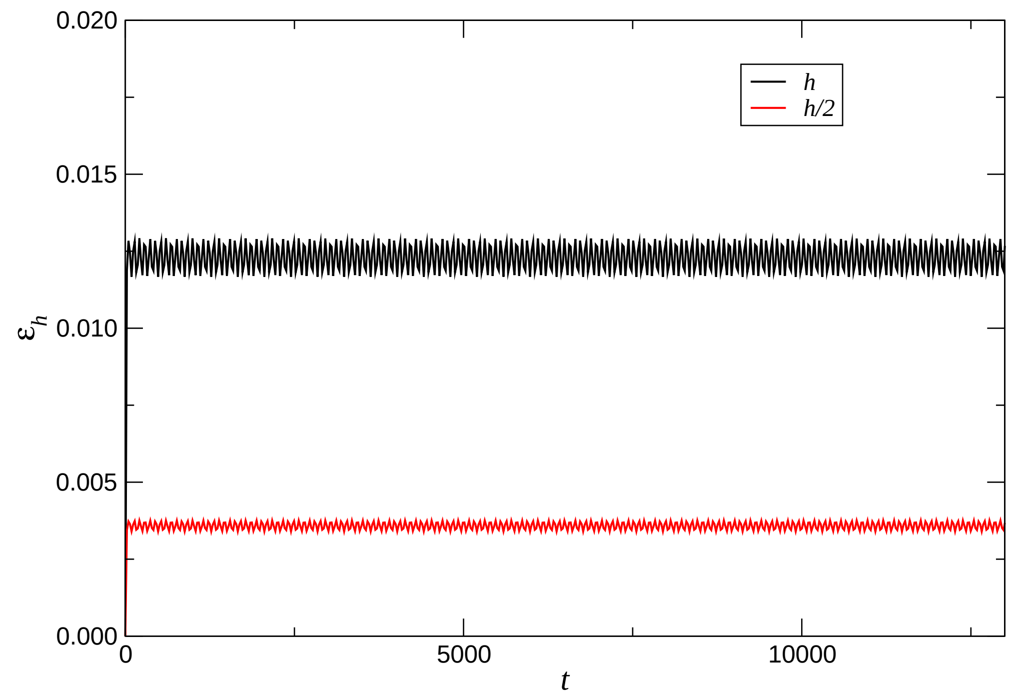
Numerical Experiments

- Boosted BH background:
  - $\beta = -0.75$;
  - $Q = \log_2 \frac{||v_h - u_{\rm exa}||}{||v_{h/2} - u_{\rm exa}||}$;
  - Cylindrical 256×512 $(\rho, z) \in [0, 10] \times [-10, 10]$;
  - Spherical 128×384 $(r, \theta) \in [2, 4] \times [0, \pi]$;
  - 4RK; $\frac{k}{h} = 3/4$;
  - dissip: 0.02.
- Go to animations



Numerical Experiments

- Minkowski background:
  - No boost ($\beta = 0$);
  - $\varepsilon_h = \|v_h - u_{\rm exa}\|_h$;
  - Cylindrical 90×170 $(\rho, z) \in [0, 10] \times [-10, 10]$;
  - Spherical 50×68 $(r, \theta) \in [1, 5] \times [0, \pi]$;
  - dissip: 0.02.
- The interpolation between the grids does not introduce growth in the error.



Conclusions

- Cubical excision has severe limitations.
- Excising the excisable can only help.
- Overlapping grids can be adapted to the geometry of the problem; allow for moving boundaries; closer to the continuum problem.
- Experiments with axisymmetric scalar field in boosted black hole background are very encouraging.
- Currently working on rotating case, higher order accuracy, dynamical tracking of excisable region, etc.
- Our belief is that if one cannot evolve a scalar field, then one cannot evolve a black hole.