Evolution of Kaluza-Klein bubbles

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Outline

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- Initial data with negative mass
- Numerical evolution
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Motivation

Witten's proof of the positive mass theorem requires the existence of certain asymptotically constant spinors. Topological obstructions. In five-dimensional Kaluza-Klein theory, space is asymptotically $\mathbb{R}^3 \times S^1$. Witten's proof is not applicable in this situation.

Questions:

- Are there negative mass configurations?
- Cosmic censorship valid?

Motivation

Negative mass configurations have been found (bubble spacetimes):

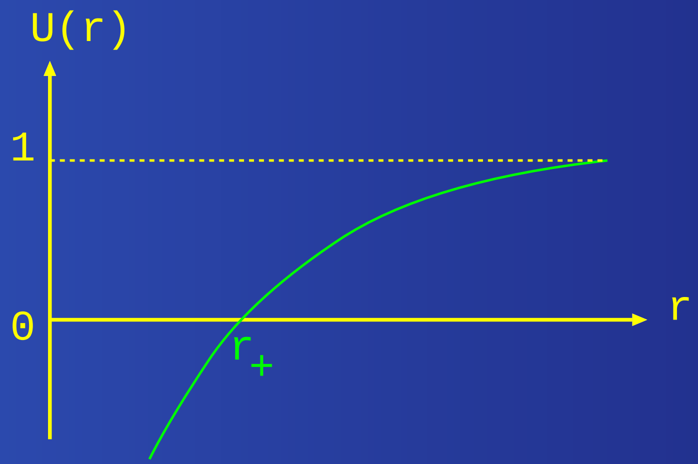
- Brill and Pfister, 1989: Solutions to the 5D vacuum constraints with negative mass.
- Brill and Horowitz, 1991: Generalization to include gauge fields
- Corley and Jacobson, 1994: Initial dynamic seems to indicate that bubble does not collapse to form a singularity
- Shinkai and Shiromizu, 2000: Numerical evolution: bubbles expand, found indication for the formation of a naked singularity

Initial data with negative mass

Consider the metric (z parametrizes the extra dimension)

$$ds^2 = -dt^2 + U(r)dz^2 + \frac{dr^2}{U(r)} + r^2 d\Omega^2,$$

where $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ and $U(r)$ is a smooth function that is zero for some $r = r_+ > 0$, satisfies $U'(r_+) > 0$, is everywhere positive for $r > r_+$ and converges to one as $r \to \infty$.



Initial data with negative mass

Consider the metric (z parametrizes the extra dimension)

$$ds^2 = -dt^2 + U(r)dz^2 + \frac{dr^2}{U(r)} + r^2 d\Omega^2.$$

What happens at $r = r_+$? New coordinate $R(r) = \int_{r_+}^{r} U(s)^{-1/2} ds$:

$$ds^2 = -dt^2 + U(r(R))dz^2 + dR^2 + r(R)^2 d\Omega^2,$$

where $U(r(R)) = (RU'(r_+)/2)^2 + O(R^4)$.
So if z has period $P = 4\pi/U'(r_+)$, the resulting spacetime $\{t, z, r \geq r_+, \vartheta, \varphi\}$ constitutes a regular manifold with the topology $\mathbb{R} \times \mathbb{R}^2 \times S^2$. By definition the bubble is located where the circumference of the extra dimension shrinks to zero, that is, at $r = r_+$.

Initial data with negative mass

Additionally, consider a $\mathbf{U(1)}$ gauge field of the form

$$A_\mu dx^\mu = k(r_+^{-n} - r^{-n})dz,$$

with k a constant and $n \in \{2, 3, 4, ...\}$.

Time-symmetric initial data satisfy the Hamiltonian constraints if

$$U(r) = 1 - \frac{m}{r} + \frac{b}{r^2} + \frac{\tilde{k}^2}{r^{2n}},$$

where m, b are free constants and $\tilde{k}$ is related to k.

The parameter m is related to the (generalized) ADM mass via $M_{ADM} = m/4$. Can be negative!!!

Initial data with negative mass

What happens to the time evolution of the negative bubble configurations? Initial acceleration of the bubble area $A = 4\pi r_+^2$ with respect to proper time:

$$\ddot{A} = 8\pi \left[1 - m - \frac{4\tilde{k}^2}{3}(n-1)(n-2) \right] \qquad (r_+ = 1).$$

Several possibilities:

- $n = 2$ (Corley and Jacobson): In this case we see that negative mass bubble will start to expand; if it continues to expand the formation of a naked singularity seems unlikely.
- $n > 2$: For k large enough one can have negative mass bubbles which start out collapsing. Formation of a naked singularity?

Numerical evolution

Parametrization of the metric:

$$ds^2 = -e^{2d}dt^2 + e^{2a}dR^2 + \frac{R^2}{r_+^2 + R^2}e^{2b}dz^2 + (r_+^2 + R^2)e^{2c}d\Omega^2,$$

where d, a, b, c are functions of t and R only.

Boundary conditions:

- At $R = \infty$: $d = a = b = c = 0$.
- At $R = 0$, require that $d' = a' = b' = c' = 0$ and that $a(t,0) - b(t,0) = const$.

Gauge condition: $d = a + \lambda(b + 2c)$, with λ a free parameter.

When $\lambda = 1$ this is similar to a densitized lapse gauge condition, when $\lambda = 0$ the resulting system has constant characteristic speeds.

Numerical evolution

Gauge field: $A_\mu dx^\mu = \gamma(t,R)dz$, $\gamma(t,0)=0$
(it turns out that nontrivial t and R components lead to a Coulomb-like electric field which diverges at the bubble).

Evolution equations (resulting from the equations $R^R_R = (\lambda+1)G^t_t$, $R^z_z = 0$, $R^\vartheta_\vartheta = 0$) have the form of a system of nonlinear wave equations

$$\ddot{u} = e^{2\lambda(b+2c)} D^{-1} \partial_R (D u') + f(u, u', \dot{u}),$$

where $u = (d+2c, b+2c, c, \gamma)^T$, and $D = diag(R^\lambda, R^{\lambda+2}, R, 1)$.
Constraints: Hamiltonian and momentum. One can show that they propagate by virtue of the regularity conditions at $R=0$ and provided that appropriate boundary conditions are given at the outer boundary.

Numerical evolution

Discretization: What to do at $R = 0$?
Consider the model problem (spherical solutions of the n-dimensional wave equation)

$$\ddot{u} = R^{-(n-1)} \partial_R (R^{n-1} u'), \qquad R > 0, t > 0.$$

Regularity condition at $R = 0$: $u'(t, 0) = 0$.
Energy conservation (or estimate if lower terms are present):

$$E = \frac{1}{2} \int_0^\infty (\dot{u}^2 + u'^2) R^{n-1} dR.$$

Idea: Descretize the system in space such that a discrete version of the energy is preserved. In particular, this implies numerical stability!

Numerical evolution

First order system: $T = \dot{u}$, $X = u'$. For $R > 0$

$$\dot{T} = R^{1-n}\partial_R(R^{n-1}X)$$
$$\dot{X} = \partial_R T$$

For $R = 0$

$$\dot{T} = n\partial_R X$$
$$\dot{X} = 0$$

Numerical evolution

First order system: $T = \dot{u}$, $X = u'$. For $R > 0$ ($R_j = j\Delta R$, $j = 1, 2, \ldots$)

$$\begin{array}{ll} \dot{T} = R^{1-n}\partial_R(R^{n-1}X) & \dot{T}_j = R_j^{1-n}D_j(R^{n-1}X) \\ \dot{X} = \partial_R T & \dot{X}_j = D_j T \end{array}$$

For $R = 0$ ($j = 0$)

$$\begin{array}{ll} \dot{T} = n\partial_R X & \dot{T}_0 = n(D_+X)_0 \\ \dot{X} = 0 & \dot{X}_0 = 0 \end{array}$$

where $D_jT = (T_{j+1} - T_{j-1})/2\Delta R$, $(D_+T)_j = (T_{j+1} - T_j)/\Delta R$.

Numerical evolution

First order system: $T = \dot{u}$, $X = u'$. For $R > 0$ ($R_j = j\Delta R$, $j = 1, 2, \ldots$)

$$\dot{T} = R^{1-n}\partial_R(R^{n-1}X) \qquad \dot{T}_j = R_j^{1-n}D_j(R^{n-1}X)$$

$$\dot{X} = \partial_R T \qquad \dot{X}_j = D_j T$$

For $R = 0$ ($j = 0$)

$$\dot{T} = n\partial_R X \qquad \dot{T}_0 = n(D_+X)_0$$

$$\dot{X} = 0 \qquad \dot{X}_0 = 0$$

where $D_jT = (T_{j+1} - T_{j-1})/2\Delta R$, $(D_+T)_j = (T_{j+1} - T_j)/\Delta R$.

Preserved energy: $E = \frac{\Delta R}{2}\sum_{j=1}^{\infty}(T_j^2 + X_j^2)R_j^{n-1} + \frac{\Delta R}{4n}T_0^2R_1^{n-1}$.

Numerical evolution

Other tricks:

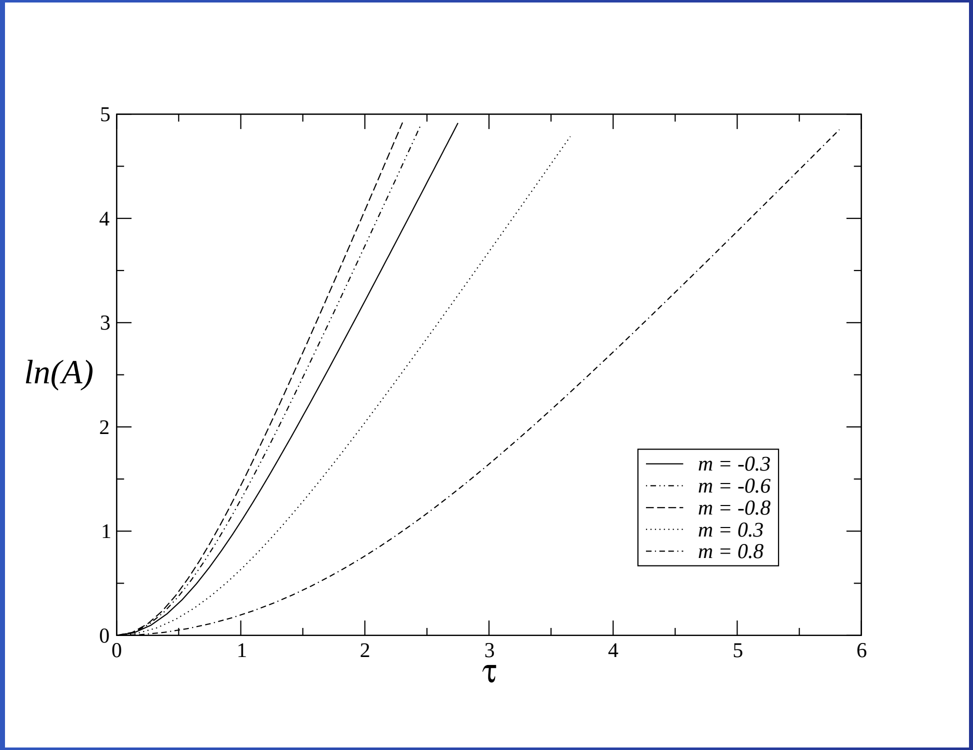
- Constraint-preserving boundary conditions (Calabrese, Lehner, Tiglio) at the outer boundary.
- Nonuniform radial coordinate to improve the resolution near the bubble:

$$R(x) = \frac{x}{1 - k\frac{x}{X_{out}}}\,, \qquad dR = \frac{dx}{\left(1 - k\frac{x}{X_{out}}\right)^2}\,, \qquad 0 \le k < 1$$

Results

A. Initially expanding case

Set the gauge field to zero, choose the mass small enough such that $\ddot{A} = 8\pi(1-m) > 0$.



Empirically:

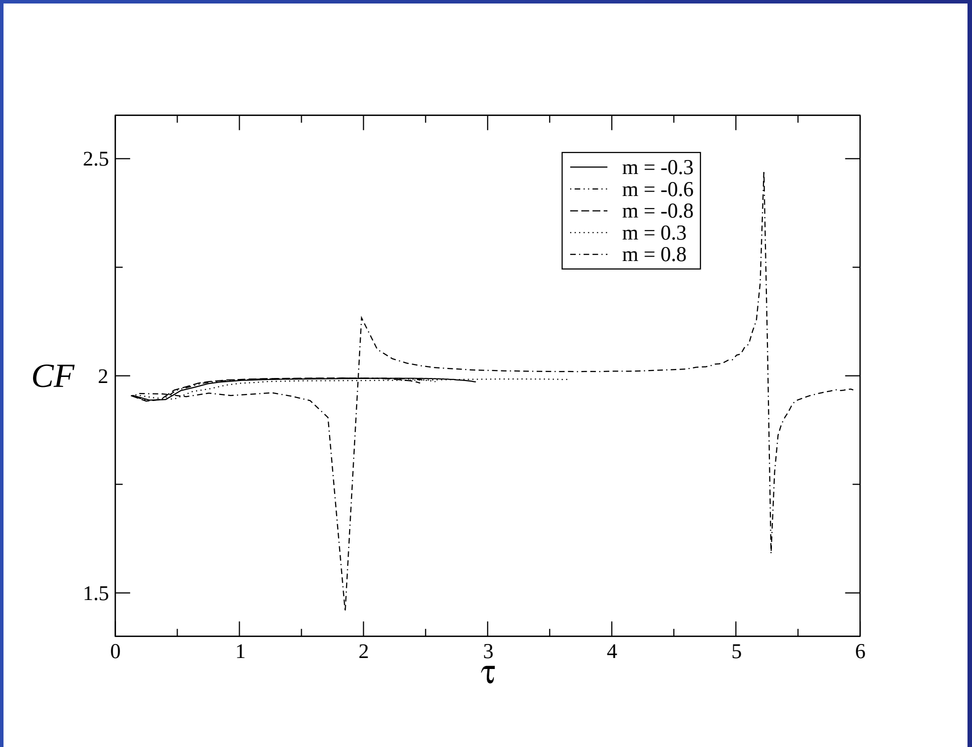
$$\frac{\dot{A}}{A} \approx \frac{2-m}{r_+(\tau=0)}.$$

(OK for Witten bubble which has $m = 0$)

Results

A. Initially expanding case

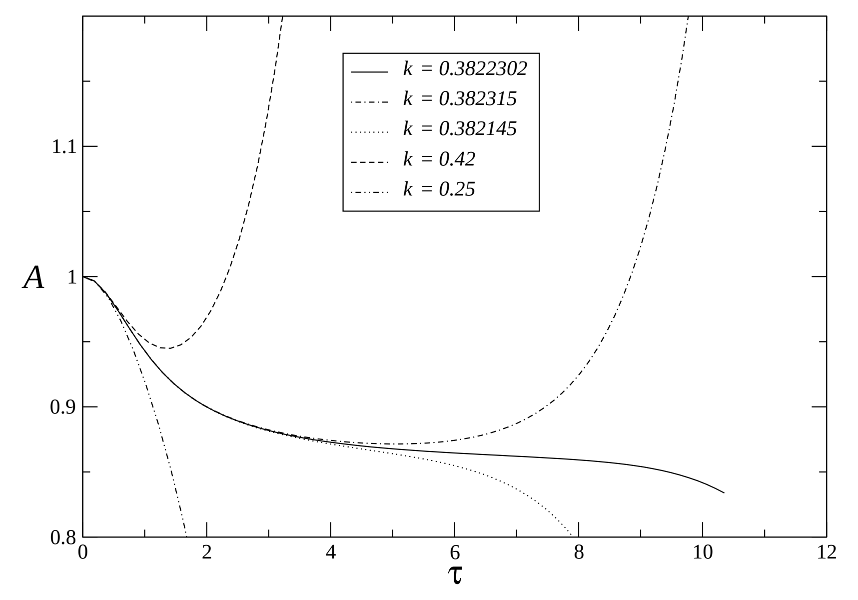
Convergence factor $CF = log_2(\|A(4\Delta) - A(2\Delta)\|/\|A(2\Delta) - A(\Delta)\|)$ vs. proper time. Here, $\Delta = 3.75 \times 10^{-3}$



Results

B. Initially collapsing case

Choose $n = 2$ so that once again the initial acceleration of the bubble is given by $\ddot{A} = 8\pi(1 - m) < 0$. Now turn on the gauge field.



Depending on the strength of the field the bubble collapses to a black string or there is a turning point and the bubble starts expanding.

Results

B. Initially collapsing case

As one fine-tunes the value of k the numerical solution seems to approach a static solution. In fact there exists a family of static solutions which is given by:

$$ds^2 = -V(r)dt^2 + \frac{V(r)}{U(r)}dr^2 + \frac{U(r)}{V(r)^2}dz^2 + r^2V(r)d\Omega^2,$$

$$A_\mu dx^\mu = \pm\frac{1}{2}\sqrt{3\left(\frac{r_+}{r_-} - 1\right)}\frac{dz}{V(r)},$$

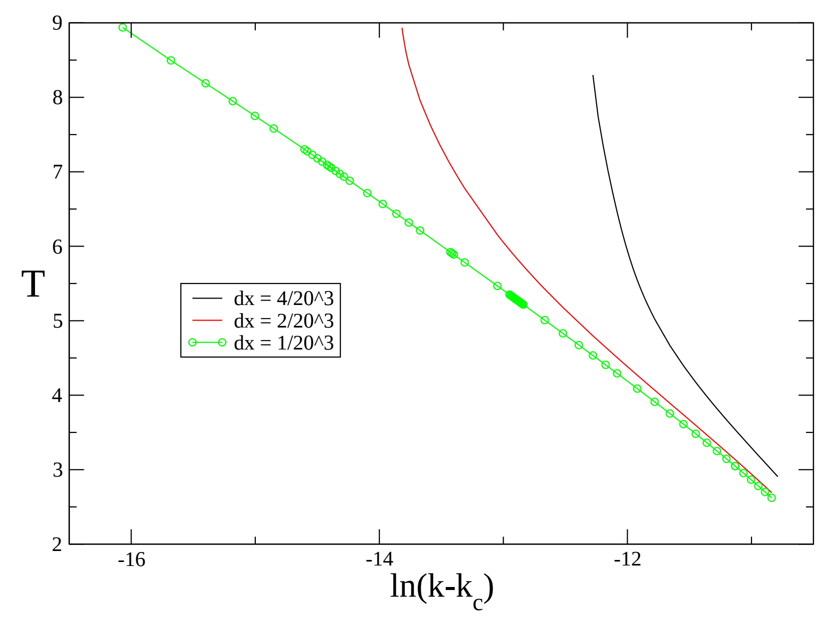
where $V(r) = 1 - r_-/r$ and $U(r) = 1 - r_+/r$.

Parameters r_- and r_+ $(> r_-)$ are related to period of the z coordinate and to ADM mass via $P = 4\pi r_+(1 - r_-/r_+)^{3/2}$ and $M_{ADM} = r_+/4$.

Results

B. Initially collapsing case

Critical phenomena?



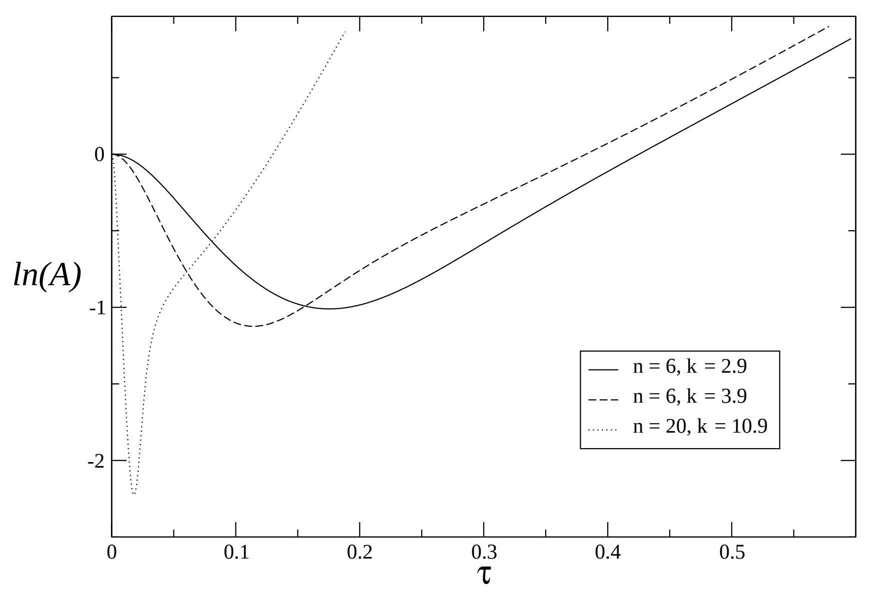
$$T = -\gamma \ln |k - k_c|.$$

$$\gamma \approx 1.21.$$

Results

C. Collapsing negative mass case

We now choose $n > 2$ and consider initial data with negative mass ($m = -0.1$) with $\ddot{A} < 0$ initially.



Apparently no formation of naked singularities!

Conclusions

- Spherically ($SO(3)$) symmetric, homogeneous sector in 5D is much richer than spherically symmetric sector in 4D gravity: Nontrivial dynamics, no Birkhoff theorem.
- In particular, there is no positive mass theorem. Configurations with negative mass do exist.
- We have found no formations of naked singularities (but clearly this is not a proof!).
- Use of a symmetric hyperbolic system of evolution equations and a careful discretization near $R = 0$ seems to make a difference!
- Collapse of a bubble to a black string needs to be investigated more carefully. Formation of apparent horizons!