

Evolution Systems for Nonlinear Perturbations of Background Geometries

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Index

1. Motivation and Main Ideas
2. Aims
3. Introductory Example
4. 3+1 Formulations
 - (a) First construction: ADM-type systems
 - (b) Second construction: Hyperbolic system
5. Numerical Implementation

Motivation and main ideas

Some Facts:

- For some physical systems, like the two-body system, the broad features of the dynamics are *approximately* known through semianalytical approaches (perturbation theory, PPN approximations).
- For numerical simulations, the structure of the terms in the non-principal part of the equations can also affect the stability of the computations.

Motivation and main ideas

Our proposal:

- The introduction of a (3D) background geometry as a *guiding* structure encompassing either the a priori knowledge about the physical system under study or an idealized situation “close” to the physical phenomena we want to simulate.
 - ➔ The true dynamics is seen as a “correction” on the background.
 - ➔ A similar philosophy has been used to study non-linear radial oscillations of neutron stars (Sperhake, Papadopoulos, and Andersson [astro-ph/0110487](#)).

Key Point: The choice of an adequate background.

Aims

- ◆ To gain accuracy in the computation for systems that do not deviate “significantly” from a certain background.
- ◆ To improve the stability of the numerical codes by taking care of the structure of the lower order terms.

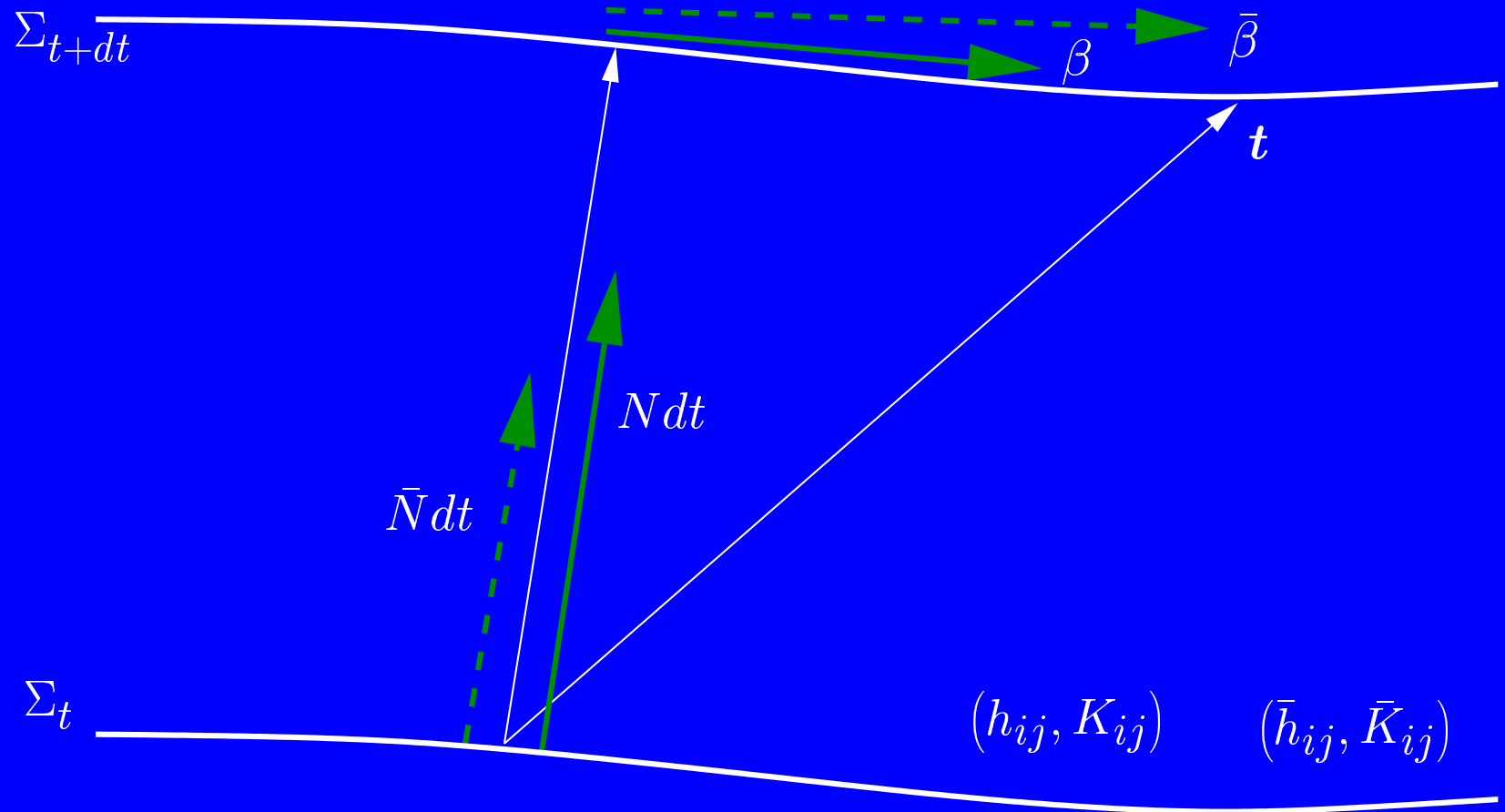
Introductory Example

[Hawking & Ellis, C.U.P., chapter 7 (1973)]

- Consider two four-dimensional metrics on the spacetime manifold M : The physical one (it satisfies Einstein's equations) $g_{\mu\nu}$, and a background metric $\bar{g}_{\mu\nu}$.
- Consider as fundamental evolution variables the *densitized inverse metric differences*: $\chi^{\mu\nu} := \det(g) (g^{\mu\nu} - \bar{g}^{\mu\nu})$
- Impose the following version of the harmonic gauge: $\bar{\nabla}_{\mu} \chi^{\mu\nu} = 0$.
- We obtain the following *reduced* Einstein equations (in vacuum):

$$g^{\rho\sigma} \bar{\nabla}_{\rho} \bar{\nabla}_{\sigma} \chi^{\mu\nu} + \text{L.O.T.}^{\mu\nu} = -2\bar{R}^{\mu\nu} + \bar{g}^{\mu\nu} \bar{R}$$

3+1 Formulations



3+1 Formulations

Some useful notation:

- For any tensor quantity $A^{i\dots j\dots}$, we define $\delta A^{i\dots j\dots} := A^{i\dots j\dots} - \bar{A}^{i\dots j\dots}$.
- $\mathcal{E}^i_{jk} := \Gamma^i_{jk} - \bar{\Gamma}^i_{jk}$ → This is a tensor on Σ .
- An alternative expression for \mathcal{E}^i_{jk} :

$$\mathcal{E}^i_{jk} = \frac{1}{2} h^{il} (\Delta_{jkl} + \Delta_{kjl} - \Delta_{ljk}) ,$$

where $\Delta_{kij} := \bar{D}_k h_{ij}$.

- $h := \det(h_{ij})$ and $\hat{\partial}_t := \partial_t - \mathcal{L}_\beta$.

First construction: ADM-type systems

- Same lapse and shift: $N = \bar{N}$, $\beta^i = \bar{\beta}^i$.
- The fundamental variables are: $(\delta h_{ij}, \delta \mathcal{K}^i_j)$, where $\mathcal{K}^i_j = h^{1/2} K^i_j$.
- The evolution equation for $\delta \mathcal{K}^i_j$ is:

$$\hat{\partial}_t \delta \mathcal{K}^i_j = h^{\frac{1}{2}} h^{ik} \left\{ \bar{D}_l (N \mathcal{E}^l_{jk}) - N (\bar{D}_{(j} \mathcal{E}^l_{k)l} - \mathcal{E}^l_{jk} \mathcal{E}^m_{lm} + \mathcal{E}^l_{m(j} \mathcal{E}^m_{k)l}) - \delta (h^{-1/2} h_{kl}) \hat{\partial}_t \bar{\mathcal{K}}^l_j + \bar{S}_{jk} \right\} .$$

Second construction: Hyperbolic system

- Here we follow the same idea as in the *Einstein-Christoffel* system [Anderson & York, PRL, **82**, 4384 (1999)] :
 - First-order symmetric hyperbolic system for $(h_{ij}, K_{ij}, \text{Combination of } \Gamma^i_{jk} \text{ and } h_{ij})$.
 - Densitized lapse: $N = \alpha(t, x^i) h^{1/2}$.

Second construction: Hyperbolic system

Main Ingredients:

- The construction require different lapses: $N \neq \bar{N}$.
- Same shift: $\beta^i = \bar{\beta}^i \rightarrow$ Same evolution operator $\hat{\partial}_t$.
- We need to introduce new variables:

$$\zeta_{kij} = \mathcal{E}_{(ij)k} + h_{k(i} h^{lm} [\mathcal{E}_{|lm|j)} - \mathcal{E}_{j)lm}] \quad (\mathcal{E}_{kij} = h_{kl} \mathcal{E}^l_{ij}).$$

- We need an evolution equation for \mathcal{E}^i_{jk} :

$$\hat{\partial}_t \mathcal{E}^i_{jk} = h^{il} \{ \bar{D}_l (NK_{jk}) - 2\bar{D}_{(j} (NK_{k)l}) + 2N \mathcal{E}^m_{jk} K_{ml} \} + \hat{\partial}_t \bar{\Gamma}^i_{jk}.$$

- To get the right principal part of the equations we need to commute some derivatives:

$$\bar{D}_{[l} \bar{D}_{k]} h_{ij} = -h_{m(i} \bar{R}^m_{j)lk}.$$

Second construction: Hyperbolic system

- Principal part of the system for $(\delta h_{ij}, \delta K_{ij}, \zeta_{kij})$:

$$\left\{ \begin{array}{l} \hat{\partial}_t \delta h_{ij} \approx 0, \\ \hat{\partial}_t \delta K_{ij} \approx -N h^{kl} \bar{D}_k \zeta_{lij}, \\ \hat{\partial}_t \zeta_{kij} \approx -N \bar{D}_k \delta K_{ij}. \end{array} \right.$$

- Characteristic directions of propagation: (i) The normal to the spacelike hypersurfaces ($v = 0$), (ii) The *physical* light cones ($v = 1$).

Numerical Implementation

- We have implemented a 3D code [For the ADM-type of construction, slide 9] and studied the 1D propagation of plane waves in a flat background.
- Use of a non-trivial background. For instance, to study the evolution of a BH+Brill wave Initial Data in a BH background.

$$h_{ij}dx^i dx^j = \Psi^4 [e^{2q}(d\rho^2 + d\theta^2) + \sin^2 \theta d\varphi^2] ,$$

$$\Psi = \Psi(\rho, \theta, \varphi) , \quad q = q(\rho, \theta, \varphi) .$$