The Constraints as Evolution Equations for Numerical Relativity

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We recast all four constraint equations as evolution equations by introducing a new set of variables. We expand the system of evolution equations to include the traditional constraint; thereby guaranteeing that that the evolution scheme conserves "momenergy." We introduce a new set of "subsidiary constraints" that must be monitored throughout the evolution. They are much simpler than the original Hamiltonian and momentum constraints.

The Central Role of the Constraints in the Numerical Evolution of Gravitational Interactions

- GR is a fully constrained theory.
- The constraints are mathematically conserved by the evolution equations.
- Violation of the Hamiltonian constraint appears to be a prognosticator of numerical failure.
- Variation of Hilbert action under diffeomorphisms generate the "momenergy" conservation equations.
- Variation of the ADM action with respect to lapse and shift yield the Hamiltonian and momentum constraints.
- Conservation of "momenergy" is a consequence of Cartan's topological boundary-of-a-boundary principle.
- Preserving constraints should naturally suppress numerically-generated non-physical "momenergy."

Constrained Evolution

- Constraint Interchange Techniques
  - e.g. Centrella & Wilson (1983-86) "Planar Numerical Cosmologies." A Fully constrained algorithm.
- Constraint Simplification/Subtraction Techniques
  - e.g. Bona-Masso, BSSN (conformal ADM), BM , KST (extended Einstein-Christoffel')
- Constraint Augmentation Technique.
  - Anderson-Matzner, gr-qc/0307055
  - Hamiltonian and Momentum Constraints as Evolution Equations, gr-qc/0307007. (Bona-Masso and BSSN partially do this via momentum constraints).

ADM $\dot{g}_{ij} - \dot{K}_{ij}$ System

12 Evolution Equations

$$(\partial_t - \mathcal{L}_{\mathcal{N}})\, g_{ij} = -2NK_{ij}$$

$$(\partial_t - \mathcal{L}_{\mathcal{N}})\, K_{ij} = N\left(R_{ij} - 2K_{il}K^l_j + TrK K_{ij}\right) - \nabla_i \nabla_j N$$

4 Constraint Equations

$$\mathcal{H} = R + (TrK)^2 - Tr\left(K^2\right) - 2\rho = 0$$

$$\mathcal{H}_i = \nabla_j K^j_i - \nabla_i TrK - S_i = 0$$

Recasting the Hamiltonian Constraint as an Evolution Equation

- Perform a conformal decomposition on the 3-metric.

$$g_{ij} = e^{4\phi}\tilde{g}_{ij}, \text{ with } det\tilde{g}_{ij} = 1$$

- Rewrite the the extrinsic curvature in terms of its trace and trace-free parts.

$$K_{ij} = A_{ij} + \frac{1}{3}g_{ij}TrK \text{ with } TrA = 0$$

$$TrK = -\frac{6}{N}\left(\partial_t - \mathcal{L}_{\mathcal{N}}\right)\phi$$

This is often used to directly evolve the conformal factor; however, we treat it as a definition of K in terms of phi dot,

The EVO-Hamiltonian Constraint

$$\mathcal{H} = R - Tr\left(A^2\right) + \frac{2}{3}\left(TrK\right)^2 - 2\rho = 0$$

$$TrK = -\frac{6}{N}\left(\partial_t - \mathcal{L}_N\right)\phi$$

Thus

$$\left(\partial_t - \mathcal{L}_N\right)\phi = \pm\frac{N}{6}\sqrt{\frac{3}{2}\left(Tr\left(A^2\right) - R + 2\rho\right)}$$

The Non-Quadratic EVO-Hamiltonian Constraint

$$\mathcal{H} = R - Tr\left(A^2\right) + \frac{2}{3}\left(TrK\right)^2 - 2\rho = 0$$

$$TrK = -\frac{6}{N}\left(\partial_t - \mathcal{L}_{\mathcal{N}}\right)\phi$$

Replace only one term!

Hence

$$\left(\partial_t - \mathcal{L}_N\right)\phi = \frac{N}{4}\left(\frac{R(g) - Tr\left(A^2\right) - 2\rho}{TrK}\right)$$

Can also introduce a new variable

$$\xi = \frac{Tr(K)\phi}{N}$$

The EVO-Momentum Constraint

$$\mathcal{H}_j = \nabla_i A^i_j - \frac{2}{3}\partial_j\left(\frac{\nabla_k N^k}{N}\right) + 4\partial_j\left(\frac{\dot{\phi}}{N}\right)$$

Commute time and spatial derivatives

$$\Phi_j \equiv \partial_j \phi$$

Thus

$$\left(\partial_t - \mathcal{L}_N\right)\Phi_j = \frac{1}{6}\left(\partial_j\partial_k N^k - Tr(K)\partial_j N\right) - \frac{N}{4}\nabla_i A^i_j$$

Implementing the STANDARD ADM $\dot{g}_{ij} - \dot{K}_{ij}$ System

16 Evolution Equations

$$(\partial_t - \mathcal{L}_{\mathcal{N}})\, g_{ij} = -2NK_{ij}$$

$$(\partial_t - \mathcal{L}_{\mathcal{N}})\, K_{ij} = N\left(R_{ij} - 2K_{il}K^l_j + TrK\,K_{ij}\right) - \nabla_i\nabla_j N$$

$$(\partial_t - \mathcal{L}_N)\,\phi = \frac{N}{4}\left(\frac{R - Tr\left(A^2\right) - 2\rho}{TrK}\right)$$

$$(\partial_t - \mathcal{L}_N)\,\Phi_j = \frac{1}{6}\left(\partial_j\partial_k N^k - Tr(K)\partial_j N\right) - \frac{N}{4}\nabla_i A^i_j$$

4 Supplementary Constraint Equations

$det(g) = e^{4\phi}$ and $\Phi_j = \partial_j\phi$

Variables: $g_{ij},\ K_{ij},\ \Phi_j$ and ϕ

Auxiliary Variables: A_{ij} and TrK

Bona-Masso & BSSN

$$2d_{jik}$$

$$V_i = \frac{1}{2} g^{ik} \left(\partial_i g_{jk} - \partial_j g_{ik} \right)$$

$$\underbrace{\partial_t V_k = -2K^{ij} \left(d_{ijk} - \delta_{ik} d^l_{\ lj} \right) + K^{ij} \left(d_{kij} - \delta_{ik} d^l_{\ jl} \right)}_{\text{Can be obtained directly from momentum constraint}}$$

$$\tilde{\Gamma}^i = -\partial_j \tilde{g}^{ij}$$

Variables are closely related

$$V_i = \tilde{V}_i + 4\partial_i \phi$$

$$\tilde{V}_j = -\frac{1}{2} \tilde{g}_{ij} \tilde{\Gamma}^i$$

Implementing the Conformal Trace-free System (Very Similar to BSSN)

17 Evolution Equations

$$(\partial_t - \mathcal{L}_N)\,\tilde{g}_{ij} = -2N\tilde{A}_{ij}$$

$$(\partial_t - \mathcal{L}_N)\,\tilde{A}_{ij} = e^{-4\phi}\left[NR_{ij} - \nabla_i\nabla_j N\right]^{TF} + N\left(Tr(K)\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right)$$

$$(\partial_t - \mathcal{L}_N)\,\phi = \frac{N}{4}\left(\frac{R - Tr(A^2) - 2\rho)}{TrK}\right)$$

$$(\partial_t - \mathcal{L}_N)\,\Phi_j = \frac{1}{6}\left(\partial_j\partial_k N^k - Tr(K)\partial_j N\right) - \frac{N}{4}\nabla_i A^i_j$$

$$(\partial_t - \mathcal{L}_N)\,TrK = N\left[Tr\left(\tilde{A}^2\right) + (1/3)(TrK)^2\right] - \nabla_i\nabla^i N + 2N\rho$$

5 Supplementary Constraint Equations

$$det\,(\tilde{g}) = 1 \qquad Tr\tilde{A} = 0$$

$$\Phi_j = \partial_j\phi$$

Differences with BSSN

- The momentum constraint is used to evolve the variables Φ_j in place of $\tilde{\Gamma}^i \equiv -\partial_j \tilde{g}^{ij}$
- The conformal factor (ϕ) is evolved using the Hamiltonian constraint rather than:

$$TrK = -\frac{6}{N}\left(\partial_t - \mathcal{L}_N\right)\phi$$

Important Points

- We guarantee that the evolution scheme conserves momenergy.
- The new subsidiary constraints are much simpler that the original constraints.
- BSSN can be viewed as a natural generalization of ADM with EVO-momentum constraints
- Difficulty may be the singularities in the Hamiltonian.
- Need to investigate relative stability.
- Relative success of BSSN and Bona-Masso provides strong motivation to include EVO-Hamiltonian.

Minkowski's Message as Guiding Principle

Hermann Minkowski
22 June 1864 - 12 January 1909

``The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.''

On September 21, 1908 Hermann Minkowski began his talk at the 80th Assembly of German Natural Scientists and Physicians.

Conserving energy-momentum (momenergy) involves both the momentum (spatial; flux) and Hamiltonian constraints (temporal; density). We believe that the incorporation of the Hamiltonian constraint as a time-evolution equation is the next logical step in NR.