The Emitted Spectrum of a Strömgren Sphere

There are four types of emission from a Strömgren sphere

1) Recombination lines of hydrogen and helium.

2) Collisionally excited lines of metals. In the optical, these are mostly forbidden lines of oxygen, nitrogren, sulfur, argon, and neon. Lines of carbon, magnesium, and silicon appear in the UV.

3) There is some continuum emission associated with nebulae, due to free-free emission, free-bound emission, and two-photon emission. (The latter occurs when a $2s$ electron of hydrogen or helium decays to the $1s$ state by temporarily making an intermediate state for itself.) At most wavelengths, this is much, much fainter than the line emission (although, at radio wavelengths, the free-free and free-bound emission is significant).

4) Other emission lines due to resonance-fluorescence, di-electronic recombination, and other odd effects.

Note that the size of a Strömgren Sphere can be defined in two ways. In can be **radiation bounded** in which case, all the ionizing photons are absorbed by hydrogen. Or it can be **matter bounded**, in which case some ionizing photons escape into space simply because there are no hydrogen atoms around for them to ionize.

The Hydrogen Spectrum

Since the A values for hydrogen are $A \sim 10^6$ sec^{-1}, while the q values are 10^{-4}, collisional excitations for hydrogen only become important when

$$N_e \gtrsim \frac{A}{q} \sim 10^{10} \text{ cm}^{-3} \tag{22.01}$$

Thus, for almost all nebulae, collisions are not important for hydrogen. One might therefore write the detailed balance equation for the level populations of hydrogen as

$$N_p N_e \alpha_{nL}(T) + \sum_{n'>n}^{\infty} \sum_{L} N_{n'L'} A_{n'L',NL} = N_{nL} \sum_{n'=1}^{n-1} \sum_{L} A_{nL,n''L''} \tag{22.02}$$

where $\alpha_{nL}(T)$ is the recombination coefficient representing the rate at which recombinations occur directly into state nL. In other words, the number of recombinations directly into a level plus the number of radiative decays into that level is equal to the number of radiative decays out of the level. If (22.02) is correct, then the emission produced by any transition is

$$j_{nn'} = \frac{h\nu_{nn'}}{4\pi} \sum_{L}^{n-1} \sum_{L'=L\pm 1} N_{nL} A_{nL,n'L'} \tag{22.03}$$

where we have explicitly kept the individual L states separate, although they are degenerate in energy (and therefore summed for the total emission).

Equation (22.02) represents a set of $n \times L$ linear equations with $n \times L$ unknowns. In theory, one can choose an electron temperature and density, look up the values of α_{nL} appropriate for the temperature, and then solve for the population levels with linear algebra. Once the level populations are known, the emission strength of each hydrogen line follow through (22.03).

In practice, the level populations are only weakly dependent on N_e and T_e. (You can see this by counting the number of ways into $n = 2$. Direct recombination is only one of many different alternatives.) Consequently, the level populations, and therefore the line ratios are almost entirely determined by the Einstein A values, not N_e or T_e.

Equation (22.02) explicitly neglects two effects. The first is collisional excitations and de-excitations. For the ground state of hydrogen, $\Delta E \ll kT_e$, so free electrons don't have enough energy to cause a collisional excitation. And, while electrons in excited states can be affected by collisions, the critical density for this is $N_e \gtrsim 10^9$ cm^{-10}. Hence, in most cases, collisions can be neglected. (Actually, this is not strictly true: electrons in extremely high levels of hydrogen ($n \sim 100$) can be moved around a bit. But this is a minor effect.)

The other piece of physics missing from (22.02) is input to levels via absorptions from lower levels. Since hydrogen electrons decay to the ground state in $\sim 10^{-6}$ sec^{-1}, virtually no absorptions occur from these states. However, absorptions from the ground state can and do occur.

To understand the effects of absorption, let's first consider the cross section of the hydrogen atom to Lyman photons.

Line	Wavelength (Å)	A (sec^{-1})	a_0 (cm^2)	$\tau_0/\tau_{912\ \AA}$
Ly α	1215.67	6.26×10^8	5.90×10^{-14}	9366
Ly β	1025.72	1.67×10^8	9.46×10^{-15}	1501
Ly γ	972.54	6.82×10^7	3.29×10^{-15}	522
Ly 10	920.96	4.21×10^6	1.72×10^{-16}	27
Ly 15	915.82	1.24×10^6	5.00×10^{-17}	8
Ly 20	914.04	5.24×10^5	2.10×10^{-17}	3

Especially for the lower Lyman lines, the cross section for absorption is *much greater* than it is for an ionization. (Recall that at 912 Å, the ionization cross section is 6.30×10^{-18} cm^2.) So, if the on-the-spot approximation is valid, Ly absorptions must be extremely important.

Traditionally, nebulae are divided into two types.

1) CASE A: all Lyman line photons escape the nebula. In this case, no Lyman absorptions occur, and equation (22.02) is strictly valid. This is the optically thin case.

2) CASE B: all Lyman line photons are re-absorbed by other hydrogen atoms. In this case, all downward transitions to the ground state don't count (in the same way as ground state recombinations don't count in the on-the-spot approximation). In this optically thick case, equation (22.02) is still valid, except transitions to the $n = 1$ level are not included in the summations.

In real life, almost everything is CASE B (or close to it), at least for the lower Lyman lines.

CASE B has an interesting implication for astrophysics. According to ionization balance, every ionization results in a recombination (with recombinations to the ground level not counted). Once in the hydrogen atom, the electrons decay downward, but decays to the ground state don't count. As a result, every decay must eventually go to the $n = 2$ state (since it can't go to the $n = 1$ state). Transitions that involve hydrogen's $n = 2$ state are in the optical! Thus, the net result is every ionization produces an optical Balmer line photon. By counting the number of Balmer photons, you can count the number of ionizing photons emitted from the central star.

The equations are even more elegant. The level populations given by (22.02) depend almost exclusively on the A values of the transitions, and are only weakly dependent to N_e and T_e. Since the level populations are (more-or-less) fixed by atomic physics, then the line-ratios are (more-or-less) fixed by atomic physics. Thus, if you measure one Balmer line, you (more-or-less) know all the Balmer line strengths. In other words, by measuring Hβ alone, you can derive the number of ionizing photons coming from the central star.

Below is a table giving the line ratios for the lowest level Balmer transitions.

Temperature	5000	10,000		20,000	
N_e (cm^{-3})	10^4	10^2	10^6	10^2	10^4
$\alpha^{\rm eff}_{\rm H\beta}$	5.44	3.02	3.07	1.61	1.61
$I({\rm H}\alpha)/I({\rm H}\beta)$	3.00	2.86	2.81	2.75	2.74
$I({\rm H}\gamma)/I({\rm H}\beta)$	0.460	0.468	0.471	0.475	0.476
$I({\rm H}\epsilon)/I({\rm H}\beta)$	0.155	0.159	0.163	0.163	0.163

$\alpha^{\rm eff}_{\rm H\beta}$ in units of 10^{-14} cm^3 s^{-1}.

In addition to line ratios, the table also has a value called $\alpha^{\rm eff}_{\rm H\beta}$. Since the recombination coefficient for hydrogen is known, and since all the line ratios for hydrogen are known, it is a relatively straightforward task to combine these numbers and figure out how many recombinations eventually create an Hβ photon. This is called the "effective" recombination coefficient for Hβ. In general,

$$N_e N_p \alpha^{\rm eff}_{nn'} = \sum_{L=0}^{n-1} \sum_{L'=L\pm1} N_{nL} A_{nL,n'L'} = \frac{4\pi j_{nn'}}{h\nu_{nn'}} \qquad (22.04)$$

Lyα and Resonance Fluorescence

Under ionization balance, every ionization is balanced by a recombination, and under Case B, every hydrogen recombination eventually results in a Balmer photon (which leaves the nebula). These $n = 2$ electrons will, of course, decay to $n = 1$, but the Lyα photons that are produced are trapped — they just get re-absorbed (and re-emitted) by other neutral atoms. This can happen hundreds or even thousands of times, but eventually, something has to happen to all these Lyα photons.

Actually, there are several possible ways to get rid of Lyα. The principle ones are

1) Lyα may "leak" out of the nebula. These photons can random walk their way to the edge of the nebula, then escape, or have their wavelength shifted slightly due to the quantum mechanics of the hydrogen atom (*i.e.,* they can be emitted "in the wings" of the line).

2) Lyα may get destroyed by hitting a dust grain. There's usually some dust floating around, and if a photon hits it, the photon will be destroyed (and the grain will be heated). One dust grain can kill alot of Lyα photons.

3) A "resonance fluorescence" occurs. Actually, this doesn't happen with Lyα per se, but with Lyβ (destroying the photon before it gets to $n = 2$) or with the Lyα of He II. (Helium has the same problem as hydrogen, and has its own Case B set of equations.)

The way resonance fluorescence works is this. Let's take He II Lyα as an example. The wavelength of this Lyα transition is 303.78 Å. A permitted transition of O^{++} ($3d^3P_2 \longrightarrow 2p^{23}P_2$) is 303.80 Å. This coincidence can cause the helium Lyα photon to excite oxygen instead of a helium ion. The oxygen atom can then decay (via another route) and emit photons of various wavelengths that escape the nebula. (There are a whole bunch of permitted oxygen emission lines between 2800 Å and 3800 Å that are excited in this way.)

Resonance fluorescence is more interesting than it is useful. Unless one is doing detailed radiative transfer calculations, it can usually be ignored.