# Principles of Cosmology

Useful (and semi-useful) references:

Most cosmology textbooks discuss the basic cosmological equations for $\Lambda = 0$ cosmologies. For material with $\Lambda \neq 0$, one can look at:

Peebles, 1993, Principals of Physical Cosmology

Hogg 2000, "Distance Measures is Comology," astro-ph/9905116

Ned Wright's Cosmology Calculator:
`http://www.astro.ucla.edu/~wright/CosmoCalc.html`

The basic assumptions of Cosmology are

1) The Universe is homogeneous – every observer sees the same thing.
2) The Universe is isotropic – there is no preferred direction in the universe.

The implication of these two statements is that the universe must either be static, or have purely radial motion. (For example, if the universe rotated, then the preferred axis of rotation would violate the assumption of isotropy.)

Let's assume that the universe is dynamic, and let

$$\begin{aligned} \vec{u} &= \text{ the co-moving coordinates of an object} \\ R(t) &= \text{ the motion (expansion or contraction) of the universe} \\ \ell &= \text{ the measured distance to an object} \end{aligned}$$

From these definitions, the observed velocity of a galaxy is

$$v = \frac{d\ell}{dt} = \frac{d}{dt}(Ru) = \dot{R}u \tag{1.01}$$

Now, neither $R$ nor $u$ is observable, but $\ell$ is. So, let's substitute $\ell$ for $u$

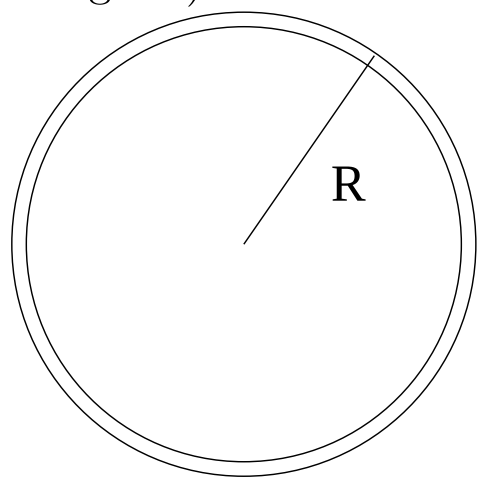
$$v = \dot{R}\left(\frac{\ell}{R}\right) = \left(\frac{\dot{R}}{R}\right)\ell = H\ell \tag{1.02}$$

Here, we have defined the variable $H(t) = \dot{R}/R$, with the units of inverse time. (The unobservable unit of scale attached to $R$ has cancelled out!) $H(t)$ is the Hubble parameter; its value today, $H_0 = \dot{R}_0/R_0$, is the Hubble Constant. $H(t)$ is the fractional rate of expansion of the universe.

## The Newtonian Universe

Consider a universe where the Newtonian laws of gravity apply. Let's pick an arbitrary position in the universe, and call it the center. Now consider the motion of a shell of material a distance $R$ from this center. ($R$ can be the extreme limit of the universe, or, if you wish, it can be just a small region.)

Recall from freshman physics that if the universe is homogeneous and isotropic, matter outside the shell will have no effect on its motion: the deceleration of the shell only depends on the matter interior to it.




So the deceleration of the shell is

$$a = \ddot{R} = -\frac{G\mathcal{M}}{R^2} \tag{1.03}$$

If we multiply each side by $\dot{R}$ and integrate over time, we get

$$\int \dot{R}\,\ddot{R}\,dt = -\int \frac{G\mathcal{M}}{R^2}\dot{R}\,dt \tag{1.04}$$

Since $\ddot{R}$ is the derivative of $\dot{R}$, and $\dot{R}$ is the derivative of $R$, both integrals are easy (of the form $\int u\,du$). The result is

$$\frac{1}{2}\dot{R}^2 - \frac{G\mathcal{M}}{R} = E \tag{1.05}$$

where $E$, the total energy, comes from the constant of integration. Note that this is nothing but energy conservation. If the potential energy of the universe is greater than its kinetic energy, $E < 0$, and eventually there will be a collapse. If the kinetic energy is greater than the potential energy, $E > 0$.

Now let's parameterize how fast the universe is decelerating. Again, since we can't measure the size of the universe, we must somehow make the units of $R$ disappear. If we take the energy conservation equation, multiply it by two, and divide by $R^2$, then

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{2G\mathcal{M}}{R^3} = \frac{2E}{R^2} \tag{1.06}$$

or, since $\ddot{R} = -G\mathcal{M}/R^2$,

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{2\ddot{R}}{R} = \frac{2E}{R^2} \tag{1.07}$$

If we multiply and divide the second term in the equation by $R/\dot{R}$ squared, then

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{2\ddot{R}}{R}\left(\frac{R}{\dot{R}}\right)^2\left(\frac{\dot{R}}{R}\right)^2 = \frac{2E}{R^2} \tag{1.08}$$

$$\left(\frac{\dot{R}}{R}\right)^2 + 2\left(\frac{\ddot{R}R}{\dot{R}^2}\right)\left(\frac{\dot{R}}{R}\right)^2 = \frac{2E}{R^2} \tag{1.09}$$

We can now define a dimensionless deceleration parameter

$$q(t) = -\frac{\ddot{R}R}{\dot{R}^2} \tag{1.10}$$

Equation (1.09) is then

$$H^2 - 2H^2 q = \frac{2E}{R^2} \tag{1.11}$$

or

$$(1 - 2q) = \frac{2E}{H^2 R^2} \tag{1.12}$$

From this definition, it is clear that if $q < 1/2$, $E > 0$ and the universe is unbound. If $q > 1/2$, $E < 0$, and we have a bound universe which will collapse. A value of $q = 1/2$ implies $E = 0$, which is a critical universe.

We can also write (1.12) in terms of the density of the universe. If we start with the acceleration term

$$\ddot{R} = -\frac{G\mathcal{M}}{R^2} \tag{1.03}$$

and substitute density for mass

$$\rho(t) = \frac{\mathcal{M}}{\frac{4}{3}\pi R^3} \tag{1.13}$$

Then

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{G\mathcal{M}}{R^2} \cdot \frac{R}{\dot{R}^2} = \frac{4}{3}\pi G\rho \left(\frac{R}{\dot{R}}\right)^2 \tag{1.14}$$

which implies that

$$\rho = \frac{3}{4\pi G} H^2 q \tag{1.15}$$

For a critical $E = 0$ universe, $q = 1/2$, so in this case

$$\rho_c = \frac{3}{8\pi G} H^2 \tag{1.16}$$

If we let $h = H_0/100$ km s$^{-1}$, then at the present time,

$$\rho_c = \frac{3}{8\pi G} H_0^2 = 1.9 \times 10^{-29} h^2 \text{ g cm}^{-3} \tag{1.17}$$

We can parameterize the universe in yet another way. If $\rho_c$ is the critical density of the universe, then the cosmological density parameter

$$\Omega(t) = \rho/\rho_c \tag{1.18}$$

Obviously, if the universe is bound, $\Omega > 1$, but if it's unbound, $\Omega < 1$. And just as obviously, $\Omega = 2q$.

## The Age of the Universe

What is the age of the universe as a function of $H_0$ and $q_0$? To compute this number, we can start with energy conservation

$$\frac{1}{2}\dot{R}^2 - \frac{G\mathcal{M}}{R} = E \tag{1.05}$$

and as before, multiply through by $2/R^2$

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{2G\mathcal{M}}{R^3} = \frac{2E}{R^2} \tag{1.06}$$

The mass of the universe (which is presumed to be constant) can be evaluated using the universe's present size and density, *i.e.*,

$$\mathcal{M} = \frac{4}{3}\pi R_0^3\,\rho_0 \tag{1.19}$$

So

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\,\rho_0\left(\frac{R_0}{R}\right)^3 = \frac{2E}{R^2} \tag{1.20}$$

First, consider an empty (Milne) universe. In this case, $\rho_0 = 0$, so equation (1.20) becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2E}{R^2} \tag{1.21}$$

or

$$\dot{R} = (2E)^{1/2} \tag{1.22}$$

This is easily integrated from $t = 0$ to the present to get

$$R(t) = (2E)^{1/2}\,t \tag{1.23}$$

The age of the universe is therefore

$$t = \frac{R}{(2E)^{1/2}} \tag{1.24}$$

This can't be evaluated, but we can substitute $\dot{R}$ for $(2E)^{1/2}$ (using 1.22). So

$$t = \frac{R}{\dot{R}} = \frac{1}{H} \tag{1.25}$$

So, for a Milne universe, the present age is $t_0 = 1/H_0$.

Now consider an Einstein-de Sitter (critical) universe. In this case, $E = 0$, and equation (1.20) becomes

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{R_0}{R}\right)^3 = 0 \tag{1.26}$$

If we multiply through by $R^2$, then the equation becomes

$$\dot{R}^2 - \frac{8}{3}\pi G\rho_0 R_0^3\, R^{-1} = 0 \tag{1.27}$$

or

$$\dot{R} = \left\{\frac{8}{3}\pi G\rho_0 R_0^3\right\}^{1/2} R^{-1/2} \tag{1.28}$$

The solution of this differential equation has the form

$$R = at^{2/3} \tag{1.29}$$

where

$$a = \left\{6\pi G\rho_0 R_0^3\right\}^{1/3} \tag{1.30}$$

(The confirmation of this is left as an exercise to the doodling student.)

Now, to calculate the age of the universe, start with $R = at^{2/3}$ and $\dot{R} = (2/3)at^{-1/3}$. We can again get rid of the unobservables by taking a ratio

$$\left(\frac{\dot{R}}{R}\right) = \frac{(2/3)at^{-1/3}}{at^{2/3}} = \frac{2}{3}\frac{1}{t} \tag{1.31}$$

So

$$t = \frac{2}{3}\left(\frac{R}{\dot{R}}\right) = \frac{2}{3}\frac{1}{H} \tag{1.32}$$

and, at the present time

$$t_0 = \frac{2}{3}\frac{1}{H_0} \tag{1.33}$$

## Cosmological Redshift

What effect does the universal expansion have on light? To see this, first consider a particle with velocity $\vec{v}$ moving past an observer at point 1 on its way to an observer at point 2. By the time it gets there, the universe has expanded; specifically, while the particle has traveled $v(t)dt$, the universe has expanded by $H\,v(t)dt$. Consequently, the observer will measure the particle's velocity to be

$$v(t+dt) = v(t) - \left(\frac{\dot{R}}{R}\right) v(t)dt \tag{1.34}$$

A bit of algebra yields

$$\frac{v(t+dt) - v(t)}{v(t)} = -\left(\frac{\dot{R}}{R}\right) dt \tag{1.35}$$

or

$$\frac{dv}{v} = -\frac{dR}{R} \tag{1.36}$$

which simply integrates to $v = R^{-1}$. The same argument applies to the frequency of a photon, yielding

$$\nu(t) = R^{-1} \tag{1.37}$$

If we now define redshift as

$$(1+z) = \frac{\nu_e}{\nu_{obs}} = \frac{\lambda_{obs}}{\lambda_e} \tag{1.38}$$

then

$$(1+z) = \frac{R_0}{R(t)} \tag{1.39}$$

## Einstein's Universe

Newtonian gravity gave the energy of the universe as

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{R_0}{R}\right)^3 = \frac{2E}{R^2} \tag{2.01}$$

In the relativisitic case, the equation is very similar

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{R_0}{R}\right)^3 = -\frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \tag{2.02}$$

Here, $k$ is a constant which defines the curvature of the universe. Note that $k$ only appears when divided by $R^2$ (which is not measureable). Thus, to simplify things, we can rescale $R$ (which is not measureable) such that $k$ is either 0, 1, or $-1$. The value $k = 1$ represents a universe with positive curvature, like a sphere; $k = -1$ reflects negative curvature, like a saddle. In the critical case, where $k = 0$, the universe is flat.

The other new variable $\Lambda$ is a Cosmological Constant. You can think of it as a pressure term which supplies a new repulsive (or attractive) force that is directly proportional to distance,

$$\ddot{R} = \Lambda R \tag{2.03}$$

In a relativistic universe, you must also take into account that the distance between two co-moving points is not simply $du$. In normal space, the distance between two points is

$$ds^2 = du^2 = dx^2 + dy^2 + dz^2 \tag{2.04}$$

The total distance, $s$ is then found by integrating along the path. In space-time, the interval between these two points is

$$ds^2 = c^2 dt^2 - du^2 \tag{2.05}$$

where the path in space *and in time* must be integrated. Finally, in the case of an expanding universe, the spatial distance between two points is $Rdu$, where $R$ is the size of the universe at the time of the measurement. So the distance between two points is

$$ds^2 = c^2 dt^2 - R^2 du^2 \tag{2.06}$$

This is called the Robertson-Walker metric. Note that for light (which travels at the speed of light), $ds = 0$.

An additional complication comes from the fact that space-time is not necessarily flat. In Cartesian coordinates

$$du^2 = dx^2 + dy^2 + dz^2 \tag{2.07}$$

which, in spherical coordinates is

$$du^2 = d\xi^2 + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2 \tag{2.08}$$

But this assumes space is flat. If space is curved, then

$$du^2 = \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2 \tag{2.09}$$

is the correct expression for distance. Note that if $k = 0$, the equation reduces to normal spherical coordinates. However, if $k = 1$, then space is elliptical, with $E < 0$, and $q > 1/2$. Conversely, if $k = -1$, then we are in the hyperbolic space, with $E > 0$ and $q < 1/2$.

## Solutions for the Friedmann $\Lambda = 0$ Universe

The first Einstein Field equation is

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{R_0}{R}\right)^3 = -\frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \qquad (2.02)$$

Now let's take $\Lambda = 0$. If $k = 0$, then space is Euclidean, and (2.02) reduces to

$$\dot{R} = \left\{\frac{8}{3}\pi G\rho_0 R_0^3\right\}^{1/2} R^{-1/2} \qquad (2.10)$$

As before, the solution to this equation is the Einstein-de Sitter universe

$$R = \left\{6\pi G\rho_0 R_0^3\right\}^{1/3} t^{2/3} \qquad (2.11)$$

If $k \neq 0$, however, the solution to the field equation cannot be written in closed form. In other words, the solution to

$$\dot{R}^2 - \frac{8}{3}\pi G\rho_0 R_0^3 R^{-1} = -kc^2 \qquad (2.12)$$

must be written parametrically. Let us define an intermediate variable, $\theta$. In terms of $\theta$, the solution to (2.12) is

| $k = 1$ (bound) | $k = -1$ (unbound) | |
|---|---|---|
| $R(\theta) = a(1 - \cos\theta)$ | $R(\theta) = a(\cosh\theta - 1)$ | |
| $t(\theta) = \frac{a}{c}(\theta - \sin\theta)$ | $t(\theta) = \frac{a}{c}(\sinh\theta - \theta)$ | (2.13) |

where

$$a = \frac{4\pi G\rho_0}{3c^2} R_0^3 \qquad (2.14)$$

Although this parametric form may seem awkward, it is very convenient for cosmological calcuations. For example, in a closed universe,

$$R = a(1 - \cos\theta) \qquad dR = a \sin\theta \, d\theta$$

$$t = \tfrac{a}{c}(\theta - \sin\theta) \qquad dt = \tfrac{a}{c}(1 - \cos\theta) \, d\theta$$

So the Hubble parameter, in terms of $\theta$, is

$$\begin{aligned} H = \frac{\dot{R}}{R} = \frac{dR}{dt} \Big/ R &= \frac{a \sin\theta}{\frac{a}{c}(1 - \cos\theta)} \div a(1 - \cos\theta) \\ &= \frac{c}{a} \frac{\sin\theta}{(1 - \cos\theta)^2} \end{aligned} \tag{2.15}$$

Similarly, after a bit of math, one can derive all the following relations:

| $\underline{k=1 \text{ (bound)}}$ | $\underline{k=-1 \text{ (unbound)}}$ | |
|---|---|---|
| $R = a(1-\cos\theta)$ | $R = a(\cosh\theta - 1)$ | (2.16) |
| $dR = a\sin\theta\,d\theta$ | $dR = a\sinh\theta\,d\theta$ | (2.17) |
| $t = \frac{a}{c}(\theta - \sin\theta)$ | $t = \frac{a}{c}(\sinh\theta - \theta)$ | (2.18) |
| $dt = \frac{a}{c}(1-\cos\theta)\,d\theta$ | $dt = \frac{a}{c}(\cosh\theta - 1)\,d\theta$ | (2.19) |
| $H = \frac{c}{a}\frac{\sin\theta}{(1-\cos\theta)^2}$ | $H = \frac{c}{a}\frac{\sinh\theta}{(\cosh\theta-1)^2}$ | (2.20) |
| $q = \frac{\Omega}{2} = \frac{1-\cos\theta}{\sin^2\theta}$ | $q = \frac{\Omega}{2} = \frac{\cosh\theta-1}{\sinh^2\theta}$ | (2.21) |
| $\cos\theta = \frac{2-\Omega}{\Omega}$ | $\cosh\theta = \frac{2-\Omega}{\Omega}$ | (2.22) |
| $(1+z) = \frac{1-\cos\theta_0}{1-\cos\theta}$ | $(1+z) = \frac{\cosh\theta_0 - 1}{\cosh\theta - 1}$ | (2.23) |
| $\cos\theta = \frac{z+\cos\theta_0}{(1+z)}$ | $\cosh\theta = \frac{z+\cosh\theta_0}{(1+z)}$ | (2.24) |

## Cosmological Proper Distance

There are several ways one can measure distance in astronomy. One can compare an observed flux to an emitted flux, and use the $1/r^2$ law: such a distance is called a "luminosity distance." Similarly, one can measure the angular size of an object, and compare it to its true physical size; this would be an "angular size distance." If the transverse velocity of an object is known, then its "proper motion" distance can be found by observing its motion on the sky. Finally, and most basically, there is the "parallax distance" to an object.

In cosmology, each of the above distances has a different dependence on $H_0$, $\Omega_0$, and $z$. Before considering them, however, let's calculate the **proper distance** a ray of light covers in going from an object at co-moving coordinate $u$ emitted at time $t_1$, to an observer at co-moving coordinate $u = 0$ at time $t_0$. To do this, we start with the Robertson-Walker metric

$$\begin{aligned} ds^2 &= c^2 dt^2 - R^2 du^2 \\ &= c^2 dt^2 - R^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2 \right\} \end{aligned} \quad (2.25)$$

For light travel, $ds = 0$, and, since the light path is purely radial, $d\theta = d\phi = 0$. So the Robertson-Walker metric simplifies to

$$c^2 dt^2 = R^2 \frac{d\xi^2}{1 - k\xi^2} \quad (2.26)$$

or, if we take the square root of both sides and move $R$ to the left-hand side of the equation,

$$\frac{c}{R} dt = \frac{d\xi}{(1 - k\xi^2)^{1/2}} \quad (2.27)$$

The proper distance between $(u, t_1)$ and $(0, t_0)$ is therefore given by

$$\int_{t_0}^{t_1} \frac{c}{R} dt = \int_u^0 \frac{d\xi}{(1 - k\xi^2)^{1/2}} \tag{2.28}$$

Now consider an Einstein-de Sitter (flat) universe. If $k = 0$, then the right-hand side of the equation is trivial. Furthermore, if we use

$$R = at^{2/3} \tag{1.29}$$

the left-hand side is only slightly harder. Therefore (2.28) reduces to

$$\frac{3c}{a} t_0^{1/3} - \frac{3c}{a} t_1^{1/3} = u \tag{2.29}$$

Now recall that the definition of redshift (1.39)

$$(1 + z) = \frac{R_0}{R_1} = \frac{at_0^{2/3}}{at_1^{2/3}} = \left(\frac{t_0}{t_1}\right)^{2/3} \tag{2.30}$$

With this substitution, (2.29) becomes

$$u = \frac{3c}{a} t_0^{1/3} \left\{1 - (1 + z)^{-1/2}\right\} \tag{2.31}$$

Finally, note that $\dot{R} = (2/3)\, at^{-1/3}$, so

$$u = \frac{2c}{\dot{R}_0} \left\{1 - (1 + z)^{-1/2}\right\} \tag{2.32}$$

and, through the definition of $H_0 = \dot{R}_0 / R_0$,

$$d_p = R_0 u = \frac{2c}{H_0} \left\{1 - (1 + z)^{-1/2}\right\} \tag{2.33}$$

Of course, the above equation is only good in the $k = 0$ case. In the general case for a $\Lambda = 0$ universe,

$$d_p = \frac{2c}{H_0 \Omega_0^2 (1 + z)} \left\{\Omega_0 z + (\Omega_0 - 2) \left[(\Omega_0 z + 1)^{1/2} - 1\right]\right\} \tag{2.34}$$

## Cosmological Luminosity Distance

Most distances in astronomy are computed via the $1/r^2$ law for light. However, for objects at cosmological distances, there are additional considerations. The flux of $n$ photons per unit time emitted from a source at redshift $z$ can be expressed as

$$F = \frac{n \cdot h\nu_e}{dt_e} = \frac{\text{photons} \cdot \text{energy}}{\text{time}} \tag{2.35}$$

Meanwhile, the flux observed is

$$f = \frac{n \cdot h\nu_0}{dt_0} \cdot \frac{1}{d^2} \tag{2.36}$$

The value $d_p$ is the actual distance the photons covers, *i.e.,* the proper distance. To relate $\nu_0$, $dt_0$ to $\nu_e$, $dt_e$, first consider the definition of redshift

$$\nu_0 = \nu_e/(1+z) \tag{1.38}$$

Next, note that $dt_0 \neq dt_e$: because the source is moving, time dilation occurs, causing us to measure the source's clocks to be slow, *i.e.,*

$$t_0 = \frac{t_e}{\left\{1-(v/c)^2\right\}^{1/2}} \tag{2.37}$$

In addition, the interval between two pulses will appear longer, because the distance between us and the source is ever increasing. (A second pulse has a longer distance to travel.) This extra time is

$$\Delta t_0 = \frac{t_e(v/c)}{\left\{1-(v/c)^2\right\}^{1/2}} \tag{2.38}$$

Thus,

$$t_0 + \Delta t_0 = t_e \frac{1 + (v/c)}{\left\{1 - (v/c)^2\right\}^{1/2}} = t_e \left\{\frac{1 + (v/c)}{1 - (v/c)}\right\}^{1/2} = (1 + z) \tag{2.39}$$

or

$$dt_0 = (1 + z)dt_e \tag{2.40}$$

So, with these two additional terms, we have

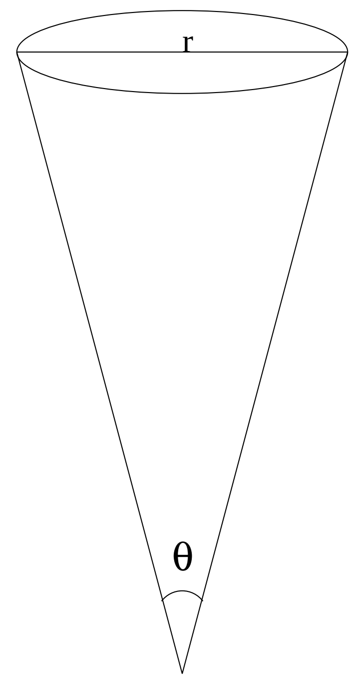
$$f = \frac{nh\nu_0}{dt_0}\frac{1}{d^2} = \frac{nh\nu_e}{dt_e}\frac{1}{(1+z)^2}\frac{1}{d^2} = \frac{F}{(1+z)^2 d_p^2} \tag{2.41}$$

The luminosity distance is therefore related to the proper distance by

$$d_L = d_p(1 + z) \tag{2.42}$$

## Angular Diameter Distance

Consider a standard galaxy with linear size $r$ at redshift $z$. Under normal Euclidean geometry, the angular size the galaxy subtends would be inversely proportional to distance. However, in an expanding relativistic universe, the calculation is a bit more complicated.




Again, let's begin with the Robertson-Walker metric

$$ds^2 = c^2 dt^2 - R^2 du^2$$
$$= c^2 dt^2 - R^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta d\phi^2 \right\} \quad (2.43)$$

If the galaxy is in the plane of the sky, then the radial distance to both sides of the galaxy is the same. Thus, $d\xi = 0$. Similarly, since both sides of the galaxy are being observed at the same time, $dt = 0$, and the separation of the two sides of the galaxy is simply $\int ds = s$. Finally, we will choose our coordinate system such that the angle subtended by the galaxy is entirely in the $\theta$ direction, so that $d\phi = 0$. Thus

$$ds^2 = -R^2 du^2 = -R^2 \xi^2 d\theta^2 \quad (2.44)$$

or, if we choose the coordinates to make $\theta$ positive

$$\theta = \frac{s}{R\xi} \quad (2.45)$$

Let's get rid of the $R$ (the size of the universe at the redshift of the galaxy) and substitute $R_0$ (the size of the universe today), using

$$(1+z) = R_0/R \tag{1.39}$$

Then

$$\theta = \frac{s}{R\xi} = \frac{s(1+z)}{R_0\xi} \tag{2.46}$$

Now recall that $R_0\xi = R_0 u$ is the proper distance. So

$$\theta = \frac{s(1+z)}{d_p} \tag{2.47}$$

In other words, the angular size distance

$$d_A = d_p(1+z)^{-1} = d_L(1+z)^{-2} \tag{2.48}$$

Note what this means for surface brightness. From dimensional analysis, the surface brightness of a galaxy is given by

$$\Sigma = \frac{f}{\theta^2} \tag{2.49}$$

where $f$ is the observed flux, and $\theta$ the observed angular size. Since $f$ is related to the intrinsic flux, $F$, via the equation for luminosity distance, and $\theta$ is related to the object's true size, $r$, via the angular size distance, then

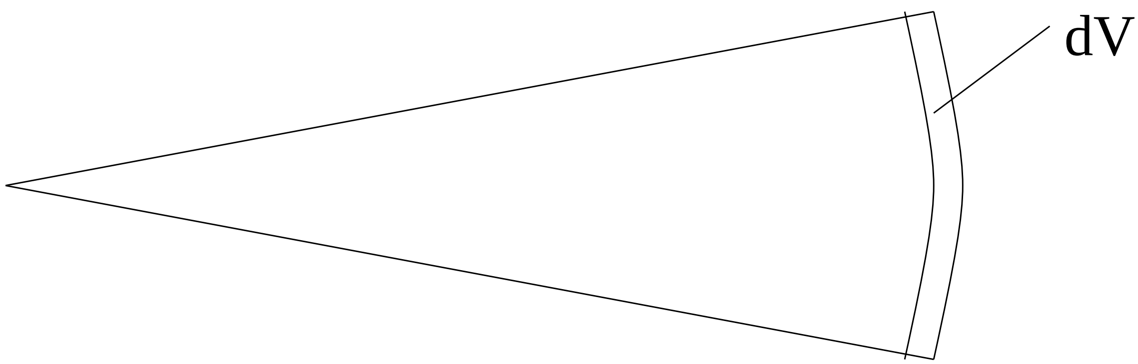
$$\Sigma = \frac{F}{d_L^2} \cdot \frac{d_A^2}{r^2} = \frac{F}{r^2}\frac{d_A^2}{d_L^2} = \frac{F}{r^2}(1+z)^{-4} \tag{2.50}$$

Objects undergo a rapid decrease in surface brightness at high redshift!

## The Cosmological Volume Element

For many calculations, you need to know how the volume of the universe changes with redshift. In general, the volume element should have the form

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi \tag{2.51}$$




The angular variables $\theta$ and $\phi$ are unaffected by cosmology and act the same as they would in Euclidean space. Since the volume shell is concentric with the earth, the radius vector, $r$, is simply the angular size distance. However, $r$ is not a very useful variable – the observable variable is $z$. Thus, a more useful way to write the volume element is

$$dV = r_A^2 \sin\theta \frac{dr}{dz} dz d\theta d\phi \tag{2.52}$$

Note that here, because $r$ is changing with $z$, use of the angular diameter distance is not appropriate for the derivative; for this. Instead, we need the proper distance at the distance of the shell, *i.e.*, $R\,du$.

To calculate the volume element, we start with the Robertson-Walker metric

$$ds^2 = cdt^2 - R^2 du^2 \tag{2.06}$$

which, for light, means $Rdu = cdt$. Since we are interested in how $R\,du$ changes with $z$, we can translate $dt$ to $dR$ using

$$dt = \left(\frac{dt}{dR}\right)\left(\frac{R}{R}\right)dR = \left(\frac{R}{\dot{R}}\right)\left(\frac{1}{R}\right)dR = \frac{1}{HR}dR \tag{2.53}$$

and convert $dR$ to $dz$ via

$$\frac{R_0}{R} = (1+z) \implies dR = -\frac{R_0}{(1+z)^2}\,dz \tag{2.54}$$

This gives us

$$R\,du = \frac{c}{HR}dR = \frac{c}{HR}\frac{R_0}{(1+z)^2}dz = \frac{c}{H(1+z)}dz \tag{2.55}$$

Now we must relate the Hubble parameter, $H$, to today's Hubble Constant. Recall that in the case of a closed universe,

$$H = \frac{c}{a}\frac{\sin\theta}{(1-\cos\theta)^2} \tag{2.20}$$

so

$$\frac{H}{H_0} = \frac{\sin\theta}{\sin\theta_0}\frac{(1-\cos\theta_0)^2}{(1-\cos\theta)^2} \tag{2.56}$$

From (2.23), the second term on the right-hand side is simply $(1+z)^2$, while the first term can be found through (2.24) to be

$$\frac{\sin\theta}{\sin\theta_0} = \frac{(\Omega_0 z+1)^{1/2}}{(1+z)} \tag{2.57}$$

This gives

$$\frac{H}{H_0} = (\Omega_0 z + 1)^{1/2} (1 + z) \tag{2.58}$$

So

$$Rdu = \frac{c}{H(1+z)} dz = \frac{cdz}{H_0 (\Omega_0 z + 1)^{1/2} (1+z)^2}$$

The volume element is therefore

$$dV = r_A^2 \frac{c}{H_0 (\Omega_0 z + 1)^{1/2} (1+z)^2} \sin\theta \, dz \, d\theta \, d\phi \tag{2.59}$$

## Cosmology with a Cosmological Constant

If there is a cosmological constant (or vacuum energy), the basic equations for distance and lookback time are somewhat changed. First let's look at the Einstein Field equation

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho = -\frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \tag{2.02}$$

In order to have a flat universe with no cosmological constant, the critical density of matter must be

$$\rho_c = \frac{3}{8\pi G}H^2 \tag{1.16}$$

If $\Lambda$ is non-zero, however, the density needed for a flat universe is decreased

$$\rho = \frac{3H^2 - \Lambda c^2}{8\pi G} \tag{3.01}$$

Because the "critical density" is no longer a well-defined quantity, what is commonly done is to keep (1.16) and define three new quantities:

$$\Omega_M = \rho_0/\rho_c = \frac{8\pi G}{3H_0^2}\rho_0 \tag{3.02}$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \tag{3.03}$$

$$\Omega_k = -\frac{kc^2}{R_0^2 H_0^2} \tag{3.04}$$

with

$$\Omega_M + \Omega_\Lambda + \Omega_k = 1 \tag{3.05}$$

Note the meaning of each term. $\Omega_M$ is the dimensionless matter density of the universe, $\Omega_\Lambda$ is a dimensionless quantity representing the "energy density" of the cosmological constant, and $\Omega_k$ is a dimensionless false density that describes the universe's departure from flatness. In an inflationary universe, $\Omega_k = 0$, and $\Omega_M + \Omega_\Lambda = 1$.

Using these variables, the equation for the expansion of the universe can be re-written and simplified.

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3}\pi G\rho_0\left(\frac{R_0}{R}\right)^3 - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \tag{2.02}$$

Multiplying through by $1/H_0^2$ yields

$$\left(\frac{R_0}{\dot{R}_0}\right)^2\left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3H_0^2}\pi G\rho_0\left(\frac{R_0}{R}\right)^3 - \left(\frac{kc^2}{R_0^2 H_0^2}\right)\left(\frac{R_0}{R}\right)^2 + \frac{\Lambda c^2}{3H_0^2} \tag{3.06}$$

or, after dividing through by $(R_0/R)^2$,

$$\left(\frac{\dot{R}}{\dot{R}_0}\right)^2 = \Omega_M \left(\frac{R_0}{R}\right) + \Omega_k + \Omega_\Lambda \left(\frac{R}{R_0}\right)^2 \tag{3.07}$$

Finally, if we substitute for $\Omega_k$ using (3.05), and let $x = R/R_0$, we get

$$\left(\frac{1}{H_0}\right)^2 \left(\frac{dx}{dt}\right)^2 = 1 + \Omega_M \left(\frac{1}{x} - 1\right) + \Omega_\Lambda \left(x^2 - 1\right) \tag{3.08}$$

This equation describes the universe as a function of three universal constants, $H_0$, $\Omega_M$, and $\Omega_\Lambda$. From this (with a little algebra), we can also derive $q_0$

$$q_0 = \frac{1}{2}\Omega_M - \Omega_\Lambda \tag{3.09}$$

## Lookback time with $\Lambda$

Using (3.08), It is fairly straightforward to calculate lookback time with $\Lambda$. Since $x = R/R_0 = (1+z)^{-1}$, (3.08) can be re-written as

$$\left(\frac{d(1+z)^{-1}}{dt}\right)^2 = H_0^2\left\{1+\Omega_M z - \Omega_\Lambda \frac{z^2+2z}{(z+1)^2}\right\}$$

$$(1+z)^{-2}\frac{dz}{dt} = H_0(1+z)^{-1}\left\{(1+z)^2(1+\Omega_M z) - \Omega_\Lambda z(2+z)\right\}^{1/2} \tag{3.10}$$

This gives

$$\Delta t = H_0^{-1}\int_0^z (1+z)^{-1}\left\{(1+z)^2(1+\Omega_M z) - \Omega_\Lambda z(2+z)\right\}^{-\frac{1}{2}} dz \tag{3.11}$$

Unfortunately, unless $\Omega_\Lambda = 0$ or $\Omega_M + \Omega_\Lambda = 1$, this integral must be evaluated numerically. Note however, that as $\Omega_\Lambda$ becomes larger, the denominator of (3.11) becomes smaller, and the lookback time increases. The additional time reconciles the expansion age of the universe with the age of its stars.

To a few percent, the total age of a universe with $\Lambda$ can be approximated by

$$t_0 \approx \frac{2}{3}H_0^{-1}\frac{\sinh^{-1}\sqrt{1-\Omega_a/\Omega_a}}{\sqrt{1-\Omega_a}} \tag{3.12}$$

where

$$\Omega_a = \Omega_M - 0.3\Omega_\Lambda + 0.3 \tag{3.13}$$

(Note that if $\Omega_a > 1$, you need to replace sinh with sin and flip the signs. Note also, that if $\Omega_a = 1$, the equation is exact.)

## Cosmological Distances with $\Lambda$

The evaluation of cosmological distances in a universe with non-zero $\Lambda$ follows closely that of the Friedmann case. For light following a radial path, the Robertson-Walker metric gives

$$\frac{c}{R}dt = \frac{d\xi}{\left(1 - k\xi^2\right)^{1/2}} \tag{2.27}$$

If we use (3.10) and manipulate the left-hand side of this equation, we get

$$\begin{aligned}
\int_{t_0}^{t_1} \frac{c}{R}dt &= \int_{t_0}^{t_1} \frac{c}{R}\,\frac{R_0}{R_0}\,\frac{dt}{dz}\,\frac{dz}{dt}\,dt \\
&= \int_0^z \frac{c}{R_0}\frac{R_0}{R}\frac{dt}{dz}dz \\
&= \int_0^z \frac{c}{H_0 R_0}\left\{(1+z)^2\left(1+\Omega_M z\right) - \Omega_\Lambda z(2+z)\right\}^{1/2} dz \\
&= |\Omega_k|^{1/2}\int_0^z \left\{(1+z)^2\left(1+\Omega_M z\right) - \Omega_\Lambda z(2+z)\right\}^{1/2} dz
\end{aligned} \tag{3.14}$$

Meanwhile, the right-hand side of (2.27) gives

$$\int_0^u \frac{d\xi}{\left(1 - k\xi^2\right)^{1/2}} = \sinh^{-1} u \tag{3.15}$$

for $\Omega_k < 1$ (and $\sin^{-1} u$ for $\Omega_k > 1$). Thus

$$\begin{aligned}
d_p &= R_0 u \\
&= R_0 \sinh\left\{|\Omega_k|^{1/2}\int_0^z \left\{(1+z)^2\left(1+\Omega_M z\right) - \Omega_\Lambda z(2+z)\right\}^{1/2} dz\right\}
\end{aligned} \tag{3.16}$$

So if we define

$$E = \int_0^z \left\{(1+z)^2(1+\Omega_M z) - \Omega_\Lambda z(2+z)\right\}^{-1/2} dz \tag{3.17}$$

then

$$\begin{aligned} &= \frac{c}{H_0}|\Omega_k|^{-1/2} \sinh\left\{|\Omega_k|^{1/2} E\right\} && \Omega_k < 0 \\ d_p &= \frac{c}{H_0} E && \Omega_k = 0 \\ &= \frac{c}{H_0}\Omega_k^{-1/2} \sin\left\{\Omega_k^{1/2} E\right\} && \Omega_k > 0 \end{aligned} \tag{3.18}$$