

Spectroscopic Notation

Most of the information we have about the universe has come to us via emission lines. However, before we can begin to understand how emission lines are formed, and what they tell us, we must first go over some basic notation and review some atomic physics. We will begin with spectroscopic notation.

Electrons are described with 4 quantum numbers: the principle quantum number n , the angular momentum quantum number ℓ , the magnetic quantum number, m , and the electron spin, s . No two electrons can have the same quantum numbers.

We can describe the locations of electrons in an atom as being in orbitals. Electrons which are in s shell orbitals have zero orbital angular momentum. Electrons in p shell orbitals have orbital angular momentum of 1, electrons in d shells have orbital angular momentum of 2, electrons in the f shell have angular momentum of 3, *etc.* The electron's spin also contributes angular momentum, with the values of $\pm 1/2$. Since angular momentum is a vector quantity, the total angular momentum of an atom is the sum of the angular momenta of all its electrons.

Only two electrons can reside in an s orbital at once; six electrons can be in a p orbital, 10 electrons be in a d orbital, and 14 electrons can be in an f orbital. Note that these numbers are not arbitrary. In the s orbital, electrons have no angular momentum, so only two can fit in (spin-up and spin-down). The unit angular momentum of the p orbital allows 6 electrons in, since the angular momentum vector can be pointed in any one of three directions (x , y , and z). For the d orbital, there are five independent orientations for the angular momentum vector (*i.e.*, $2x$, $2y$, $2z$, $x + y$, and $x + z$); for the f orbital, 7.

There is a relation between the quantum numbers of an atom. In the ground $n = 1$ state, only s orbitals are allowed; in the $n = 2$ state, both s and p orbitals exist; in the $n = 3$ state, you can s , p , and d orbitals. In order, the first few shells of an atom are

$$1s^2, 2s^2, 2p^6, 3s^2, 3p^6 \dots \quad (12.01)$$

Rather than specify the quantum number of every electron, it is more useful to give a single quantum number for the entire atom. Thus, spectroscopic notation uses S , P , D , and F to denote the total orbital angular momentum of all the electrons, *i.e.*,

$$\vec{L} = \sum_i \vec{\ell}_i \quad (12.02)$$

When $L = 0$, the atom is in an S state; when $L = 1$, it is in a P state, and so on. Note that since angular momentum has a direction associated with it, the individual ℓ values are added vectorially.

Similarly, spectroscopic notation just gives one number for the sum of all the electron spins, *i.e.*,

$$\vec{S} = \sum_i \vec{s}_i \quad (12.03)$$

Since electrons can only have spin-up or spin-down, a 2-electron system can only have $S = 0$ or $S = 1$.

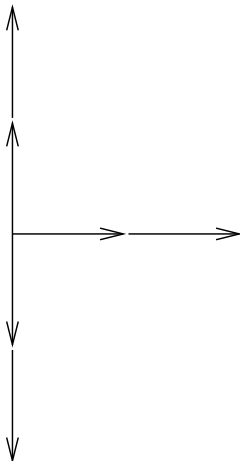
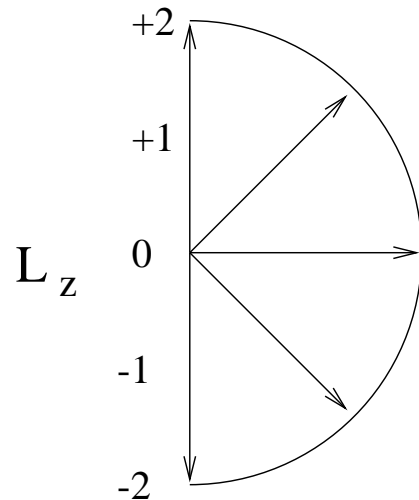
For light elements, “LS coupling” is a good rule; this simply means that the total angular momentum of an atom is the (vector) sum of all the orbital and spin angular momenta

$$\vec{J} = \vec{L} + \vec{S} \quad (12.04)$$

The state of an atom can thus be described using L , J , and S , instead of m and s for each electron. In spectroscopic notation, we write the state of an atom as $^{2S+1}L_J$, where $2S + 1$ is called the “multiplicity”, and L is called the “term”. For instance, an atom of neutral carbon in the ground state can be described as $1s^2, 2s^2, 2p^2 \ ^3P_0$.

Since L , S , and J are all quantized vectors, they can have more than one orientation in space. In addition, the z -components of these vectors are also quantized. This has several implications.

First, a single L term consists of $(2L + 1)$ states. Due to spin-orbit interactions, these states do not necessarily have the same energy.



Second, for any value of S , there are $(2S + 1)$ states. For instance, two electrons with the *same* spin can be in 3 different states (with 3 different orientations), while two electrons with opposite spins have only one possible state.

Finally, each value of J represents $(2J + 1)$ separate states. Normally, since there is no preferred direction for atoms, the energies of each of the $(2J + 1)$ states are equivalent. However, this degeneracy can be split by giving the atom a preferred direction (say, with an electric or magnetic field).

Note that when a shell is filled, the electrons occupy all possible orientations, so that the net L and S of the shell is zero. Thus, only unfilled shells contribute to the L and S of the atom. Similarly, from symmetry, a shell with all the orbitals filled except one looks and acts the same as a shell with only 1 electron, and shells that are missing two electrons are equivalent to shells that contain two electrons.

As an example, consider a 3P term. Since it is a P term, $L = 1$, and, from the multiplicity, $S = 1$. It therefore contains $(2L + 1)(2S + 1) = 9$ separate states. Since $L = 1$ and $S = 1$ can add vectorially to $J = 2, 1$, or 0 , these states are 3P_2 , which contains $2J + 1$, or 5 of the states, 3P_1 , which holds three of the states, and 3P_0 , which is a single state. The value $\omega = (2J + 1)$ is called the *statistical weight* of a level, since it gives the number of degenerate states at a single energy level.

Transition Rates

The probability of an electron in an atom making a transition from state n to state n' is given by the Einstein probability coefficients. For $n > n'$, these are

- $A_{nn'}$: the spontaneous emission coefficient
- $B_{nn'}$: the induced emission coefficient
- $B_{n'n}$: the absorption coefficient

In other words, the Einstein A value is the probability of an electron decaying from state n to state n' all by itself. The Einstein $B_{nn'}$ value deals with stimulated emission: a passing photon with an energy equivalent to the jump from n to n' can prompt (*i.e.*, stimulate) an electron decay. (This is the way lasers and masers work.) Finally, the $B_{n'n}$ values deal with the probability of absorption. All three coefficients are fundamental constants associated with atomic levels; $A_{nn'}$ has the units of inverse seconds. These coefficients are related: it can be shown (with relatively little effort), that

$$\omega_n B_{nn'} = \omega_{n'} B_{n'n} \quad (12.05)$$

and

$$A_{nn'} = \frac{2h\nu^3}{c^2} \frac{\omega_{n'}}{\omega_n} B_{n'n} \quad (12.06)$$

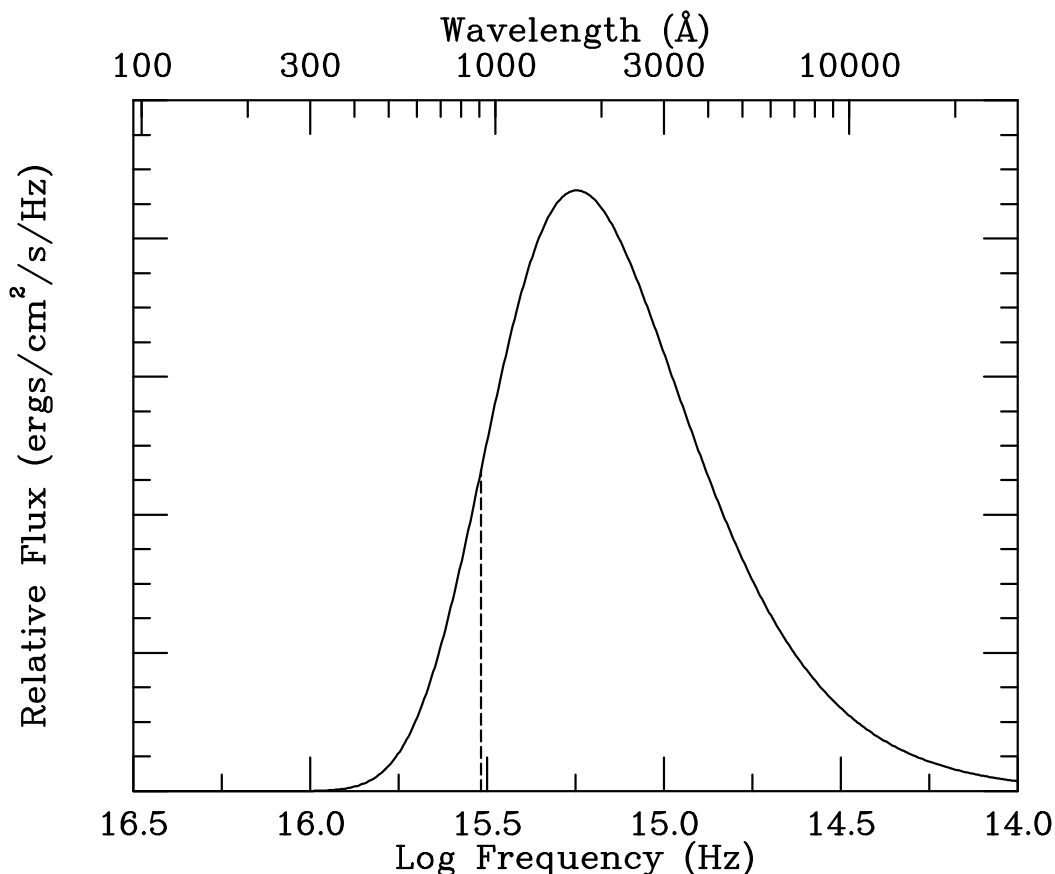
where ω is the statistical weight of the level. High A values (such as 10^8 sec^{-1}) are permitted transitions; on average, if $A = 10^8 \text{ sec}^{-1}$, it will take an electron 10^{-8} seconds to decay. Also, in general, the larger the energy difference of a permitted transition, the larger the A value, with $A \propto \nu^3$.

Note that not all transitions in an atom are equally probable. From quantum mechanics, the most probable transitions must have $\Delta n \neq 0$, $\Delta \ell \pm 1$, and $\Delta j \pm 1$ or 0 (unless j is zero to start with, in which case j must change by ± 1). There are several reasons for these rules, including conservation of angular momentum (the photon carries away 1 unit of j), and the ability of the atom to radiate via dipole radiation. If these rules are not satisfied, the transition is classically “forbidden”. In practice, that means that the A value for the transition is very low, typically $A \lesssim 1 \text{ sec}^{-1}$.

The Photo-ionization Cross Section

To understand the physics of an emission region, one needs to know two functions. The first is the hydrogen atom cross section for photo-ionization from the ground state. It takes 13.6 eV of energy to ionize hydrogen; this corresponds to a wavelength of 912 Å, and a frequency of 3.29×10^{15} Hz. At this energy, the atom's cross section to a photon is $A_0 = 6.30 \times 10^{-18}$ cm². At higher energies, the cross section decreases, roughly as $1/\nu^3$, so that for hydrogenic atoms,

$$a_\nu = \left(\frac{A_0}{Z^2} \right) \left(\frac{\nu_0}{\nu} \right)^3 \quad (12.07)$$



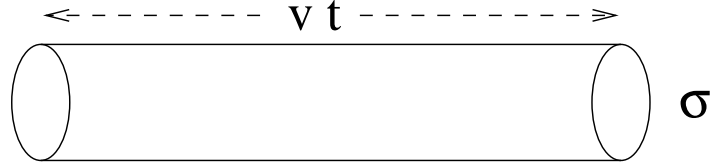
The implication is that extreme-UV light with wavelengths just shortward of 912 Å are absorbed very rapidly in the neutral hydrogen in the interstellar medium. Consequently, at these wavelengths, only a few objects in the sky are visible.

Because of ν^{-3} dependence of absorption, at shorter wavelengths (say, in the soft x-ray band), absorption by interstellar gas is much less important (although it still cannot be ignored); by the time you reach the hard x-ray and γ -ray bands, the ISM is virtually transparent. Main-sequence stars have temperatures such that only the very tail of their blackbody curves extends shortward of 13.6 eV. Consequently, the “average” cross section for photoionization in these regions is quite large.

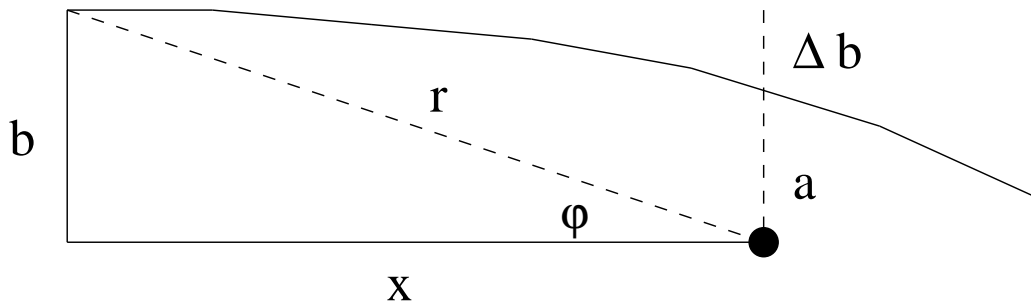
The Recombination Coefficient

The other parameter that is needed is the recombination coefficient for hydrogen, α . The rate at which free electrons will recombine into (any atomic level of) hydrogen depends on how fast the electrons are moving.

Fast moving will sweep out a larger volume of space than slow moving electrons, so $\alpha \propto v\sigma_{nL}$.



On the other hand, the cross sectional area for recombination, σ_{nL} must be inversely dependent on velocity, due to electrostatic focusing of the electrons by the protons. The calculation of this effect is straightforward:



An electron that starts out moving towards a proton with impact parameter, b , and velocity, v , feels a force perpendicular to its motion,

$$F_{\perp} = m_e \frac{dv_{\perp}}{dt} = \frac{e^2}{r^2} \sin \phi = \frac{e^2}{r^2} \frac{b}{r} = \frac{e^2 b}{r^3} \quad (12.08)$$

Integrating this once over $dt = dx/v$ gives the velocity perpendicular to the direction of original motion,

$$\begin{aligned}
v_{\perp}(x') &= \frac{e^2 b}{m_e} \int_{-\infty}^{x'} (x^2 + b^2)^{-3/2} \frac{dx}{v} \\
&= \frac{e^2 b}{m_e v} \left\{ \frac{x}{b^2 (x^2 + b^2)^{1/2}} \right\} \Bigg|_{-\infty}^{x'} \\
&= \frac{e^2}{m_e v b} \left\{ \frac{x'}{(x'^2 + b^2)^{1/2}} + 1 \right\} \tag{12.09}
\end{aligned}$$

Integrating the equation a second time gives the difference between the initial impact parameter, and the impact parameter at closest approach

$$\begin{aligned}
\Delta b &= \int_{-\infty}^0 v_{\perp}(x') dt = \frac{e^2}{m_e v b} \int_{-\infty}^0 \left\{ \frac{x'}{(x'^2 + b^2)^{1/2}} + 1 \right\} \frac{dx'}{v} \\
&= \frac{e^2}{m_e v^2 b} \left\{ (x'^2 + b^2)^{1/2} + x' \right\} \Bigg|_{-\infty}^0 = \frac{e^2}{m_e v^2} \tag{12.10}
\end{aligned}$$

Thus, the change in the cross sectional area is

$$\Delta \sigma = \pi b^2 - \pi (b - \Delta b)^2 = 2\pi b \Delta b - \pi \Delta b^2 \tag{12.11}$$

which, in the limit of large impact parameters and small deflections, means that

$$\Delta \sigma = 2\pi b \Delta b \propto v^{-2} \tag{12.12}$$

Thus, due to electrostatic focusing, the cross section for recombination is inversely proportional to the square of the electron velocity.

The two effects combined imply that $\alpha \propto v/v^2 \propto v^{-1}$. In practice, not all electrons have the same velocity: the distribution of velocities is Maxwellian. (It takes electrons in the interstellar medium only a few hours to become totally thermalized.) Since the characteristic temperature of the electrons, $T_e \propto v^2$, this means that the recombination coefficient, $\alpha \propto T_e^{-1/2}$. Thus, α is a weakly dependent on temperature. Numerically, the recombination coefficient for all levels of the hydrogen atom is

$$\begin{aligned} \alpha_A &= 6.82 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1} \quad \text{for } T_e = 5,000 \text{ K} \\ \alpha_A &= 4.18 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1} \quad \text{for } T_e = 10,000 \text{ K} \\ \alpha_A &= 2.51 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1} \quad \text{for } T_e = 20,000 \text{ K} \end{aligned}$$

Parameters of a Strömngren Sphere

Let's assume that we have a 40,000 K B-main sequence star surrounded by an interstellar medium composed entirely of hydrogen (density $N(H) = 1$ particles cm^{-3}).

How long will a neutral hydrogen atom stay neutral? The photon luminosity of a B main sequence star is

$$\int_{\nu_0}^{\infty} \frac{L_{\nu}}{h\nu} d\nu = 5 \times 10^{48} \text{ photons s}^{-1} \quad (12.13)$$

where ν_0 is the ionization frequency for hydrogen. Let's now assume all the ionizing radiation comes from the star. The photoionization rate at a distance r from the star is

$$P = N(H^0) \int_{\nu_0}^{\infty} \frac{L_{\nu}}{4\pi r^2 h\nu} a_{\nu} d\nu \quad (12.14)$$

Note the terms in the equation. The photoionization rate is proportional to the number of neutral atoms that are around to be photoionized, the number of photons available for photoionization, and the cross section for ionization. For a precise calculation, we should integrate this equation, but for now, we can just adopt an “average” value of a_{ν} . For a 40,000 K star, only the high-end tail of the blackbody function extends shortward of 912 Å, hence almost all ionizations occur with $a_{\nu} \approx 5 \times 10^{-18} \text{ cm}^2$. So

$$P = N(H^0) \int_{\nu_0}^{\infty} \frac{L_{\nu}}{4\pi r^2 h\nu} a_{\nu} d\nu \approx N(H^0) \bar{a}_{\nu} \int_{\nu_0}^{\infty} \frac{L_{\nu}}{h\nu} d\nu \quad (12.15)$$

If we now pick a position, say, 5 pc away from the central star, then we can just plug in the numbers to get $P \sim 10^{-8}$ ionizations $\text{cm}^{-3} \text{ s}^{-1}$. In other words, in a cubic centimeter of space, the time between ionizations is 10^8 seconds: a neutral hydrogen atom can stay neutral for over a year!

What is the distribution of states for hydrogen? The time it takes an atom to decay from state nL to state $n'L'$ is just one over the Einstein A value. The time it takes an atom to decay from state nL to *any* state is therefore

$$\tau_{nL} = \frac{1}{\sum_{n' < n} \sum_{L' = L \pm 1} A_{nL, n'L'}} \quad (12.16)$$

(Note that by specifying that $L' = L \pm 1$, we are neglecting the forbidden transitions.). Typical A values for permitted transitions are $10^4 \lesssim A_{nL, n'L'} \lesssim 10^8 \text{ sec}^{-1}$, so $10^{-4} \gtrsim \tau_{nL} \gtrsim 10^{-8}$ seconds.

There is one exception to this. A hydrogen atom in the 2^2S state, has no permitted way to decay down to 1^2S . Atoms in this state must either be a) collided out of the state, or b) decay via a low probability ($A = 8.23 \text{ sec}^{-1}$) two-photon emission, where the atom makes a temporary state for itself between 1^2S and 2^2S . If this occurs, the atom will be trapped in 2^2S for $\tau = 0.12$ seconds.

In any event, the time for a hydrogen atom to decay is less than a second, whereas it must wait a year before a neutral atom gets photoionized. Moreover, in the low-density region of the interstellar medium, the timescale for collisions between electrons and atoms (in order to excite an electron to a higher energy level) is much longer than the timescale to decay. As a result, essentially all the atoms will be in their ground state. The ISM is *not* in thermodynamic equilibrium!

Finally, note that the energy difference between the ground state and first excited state of hydrogen (10.2 eV, or $\lambda = 1216 \text{ \AA}$) corresponds to

$$\frac{hc}{\lambda} \sim \frac{1}{2}kT \implies T \sim 80,000 \text{ K} \quad (12.17)$$

This is much hotter than the electrons in an H II region (which, as we will see, typically have temperatures of $T \sim 10,000$ K), so collisional excitation of hydrogen from $n = 1$ to $n = 2$ is unimportant. Again, this means that all the neutral hydrogen will be in the ground state.

What fraction of hydrogen atoms will be neutral? Let ξ be the fraction of neutral hydrogen, *i.e.*, $\xi = N(H^0)/N(H)$, and again, let's consider a spot 5 pc from a 40,000 K main sequence star. As we have seen, at that location

$$P = N(H^0) \int_{\nu_0}^{\infty} \frac{L_\nu}{4\pi r^2 h\nu} a_\nu d\nu \approx N(H^0) 10^{-8} \text{ sec}^{-1} \quad (12.18)$$

For an emission region to be in equilibrium, the number of ionizations (in each cubic centimeter) must equal the number recombinations. The latter number is given by

$$R = N_e N_p \alpha_A \quad (12.19)$$

In other words, the recombination rate is proportional to the density of free electrons, the density of hydrogen atoms needing an electron, and the recombination coefficient, $\alpha_A \sim 4 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1}$. Ionization balance implies

$$N(H^0) \int_{\nu_0}^{\infty} \frac{L_\nu}{4\pi r^2 h\nu} a_\nu d\nu = N_e N_p \alpha_A \quad (12.20)$$

In a pure hydrogen nebula, $N_e = N_p$, and $N_p = (1 - \xi)N(H)$. Thus

$$\xi N(H)(10^{-8}) = (1 - \xi)^2 N(H)^2 (4 \times 10^{-13}) \quad (12.21)$$

or

$$2.5 \times 10^{-4} \xi = (1 - \xi)^2 N(H) \quad (12.22)$$

For a typical ISM value of $N(H) \sim 1$, $\xi = 4 \times 10^{-5}$. Virtually all the hydrogen is ionized!

How thick is the transition region between the ionized and neutral material? The mean free path of an ionizing photon is given by

$$\ell = \frac{1}{N(H^0)a_\nu} \quad (12.23)$$

(The larger the cross section, or higher the density of neutral atoms, the shorter the path.) By definition, in the transition region, half of the hydrogen is neutral. Thus the mean free path of an ionizing photon in this region is

$$\ell_{trans} \sim \frac{1}{0.5N(H)a_\nu} \quad (12.24)$$

For $a_\nu \sim 5 \times 10^{-18} \text{ cm}^2$ and $N(H) \sim 1$, $\ell_{trans} \approx 4 \times 10^{17} \text{ cm}$, or 0.1 pc. Thus, when the material starts to become neutral, the ionizing photons are rapidly eaten up. Considering that the entire H II region might be $\sim 10 \text{ pc}$ across, the transition region is very thin.

(Note: if the source of ionizing radiation is very hard (for instance, as in the power law continuum of an AGN), then the effective value of a_ν can be significantly smaller. This can translate into a large transition region.)

What is the size of an ionized region? To calculate the size of an emission region, let's start with the full equation for ionization balance. The ionization rate at every location in the nebula must be equal to the recombination rate at that location.

$$N(H^0) \int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} a_\nu d\nu = N_e N_p \alpha_A \quad (12.25)$$

Here, J_ν is the mean intensity of radiation, in units of energy per unit area, per unit time, per unit frequency, per unit solid angle.

(The 4π term is the implicit integration over all directions.) Note that J_ν has two components. Ionizing radiation can come from the central star, but it can also come from a diffuse component. If, somewhere in the nebula, an electron recombines from the continuum directly into the ground state, the photon emitted will have enough energy to ionize another atom.

Keeping track of the diffuse ionizing field is fairly complicated. However, for many nebulae, there is an excellent approximation. Since the cross section for recombination is inversely proportional to velocity, most recombinations will be from slow moving electrons. Those recombinations that go directly into the ground state will therefore produce a photon with $\nu \approx \nu_0$; such a photon has a very high cross section for re-absorption. Therefore, let us make the **on-the spot assumption** that **all photons produced by ground state recombinations are absorbed locally**. Any recombination to the ground state produces a photon that is immediately re-absorbed by a neighboring atom: it's as if no ground state recombination ever occurred.

(In low density nebulae with $N(H) \sim 1$, this approximation is not very good, since, as we have seen, the mean free path for an ionizing photon is

$$\ell = \frac{1}{N(H^0)a_\nu} \quad (12.23)$$

If the neutral fraction is small, $\xi \sim 10^{-4}$, then ℓ can be large. However, as the density of particles increases, the mean free path decreases, and the approximation becomes better and better.)

If we neglect the diffuse component of the ionizing radiation field, then the only ionizing photons are those that come from the central star. If each square centimeter of the star produces a monochromatic flux of πF_ν ($\text{ergs cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1}$), then any point

in the nebula sees this field, diluted by a geometrical term, and by the attenuation that has already occurred. In other words,

$$N(H^0) \int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} a_\nu d\nu = N(H^0) \int_{\nu_0}^{\infty} \frac{\pi F_\nu}{h\nu} \left(\frac{R^2}{r^2} \right) a_\nu e^{-\tau_\nu} d\nu \quad (12.26)$$

where R is the radius of the star, r is the distance to the star, and τ , the optical depth, reflects the fact that some photons from the star have already been absorbed before reaching r . Specifically, the equation for the optical depth is

$$\tau_\nu(r) = \int_0^r N(H^0)(r') a_\nu dr' \quad (12.27)$$

Since the luminosity of the star is

$$L_\nu = 4\pi R^2 (\pi F_\nu) \quad (12.28)$$

the equation of ionization balance simplifies to

$$\frac{N(H^0)}{4\pi r^2} \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} a_\nu e^{-\tau_\nu} d\nu = N_e N_p \alpha_B \quad (12.29)$$

where now we are using α_B , which is the recombination rate to all levels except the ground state of hydrogen.

Now, assume that the density of the emission region is roughly constant. We can re-write (12.29) as

$$N(H^0) \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} a_\nu e^{-\tau_\nu} d\nu = 4\pi r^2 N_e N_p \alpha_B \quad (12.30)$$

and integrate both sides over r , with \mathcal{R} representing the extent of the nebula.

$$\int_0^{\mathcal{R}} N(H^0) \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} a_\nu e^{-\tau_\nu} d\nu dr = \int_0^{\mathcal{R}} 4\pi r^2 N_e N_p \alpha_B dr \quad (12.31)$$

We can replace dr with $\frac{d\tau}{N(H^0)a_\nu}$, since

$$\tau_\nu(r) = \int_0^r N(H^0)(r')a_\nu dr' \implies \frac{d\tau}{dr} = N(H^0)a_\nu \quad (12.32)$$

At the edge of the nebula $\tau = \infty$ (by definition), so

$$\int_0^{\mathcal{R}} N(H^0) \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} a_\nu e^{-\tau_\nu} \frac{d\tau}{N(H^0)a_\nu} = \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} d\nu \int_0^{\infty} e^{-\tau_\nu} d\tau \quad (12.33)$$

or just

$$\int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} d\nu = Q(H^0) \quad (12.34)$$

In other words, the left hand side of (12.31) is just the number of ionizing photons coming from the star. (This should not be too surprising, since, one way or the other, all the photons must be eaten up.) In the case of a constant density nebula, the right hand side of (12.31) is straightforward to integrate, and, since the ionization fraction inside the nebula is extremely high, $N_p \approx N_e \approx N(H)$. So

$$Q(H^0) = \int_0^{\mathcal{R}} N_e N_p \alpha_B dV = \frac{4}{3} \pi \mathcal{R}^3 N(H)^2 \alpha_B \quad (12.35)$$

Thus, we have a very simple expression for the size of the nebula. Our 40,000 B-main sequence star will generate an H II region ~ 50 pc in radius.

Modifications Due to Helium

Photons with energies greater than 24.6 eV (504 Å) can ionize helium and, due to the ν^{-3} dependence on cross section, such photons are ~ 8 times more likely to ionize helium than hydrogen. At first glance, this would seem to mean that helium can significantly reduce the size of a Strömngren sphere. However, the energy difference between the $n = 1$ state of helium, and the $n = 2$ state is about 19.9 eV. (The exact difference depends on whether we are talking about singlet states, or triplet states, and whether the $n = 2$ term is a P or an S .) Thus, almost every recombination of a helium atom will produce another photon capable of ionizing hydrogen. Consequently, the effect of helium on the size of the Strömngren sphere is minor.

(Note: some interesting physics is associated with the He I atom. Since there are two electrons, He I has singlet levels (opposite spin electrons) and triplet levels (same spin electrons). Like hydrogen, singlet decays from 2^1P to 1^1S are no problem, but 2^1S to 1^1S is forbidden; these atoms can only decay by creating a temporary state somewhere between 2^1S and 1^1S and emitting 2 photons ($A = 51 \text{ sec}^{-1}$). When this occurs, 56% of the time, one of the photons will have an energy greater than 13.6 eV. Atoms in the triplet state, however, have a harder problem: once they get to 2^3S , the only way for them to decay is to have one electron change its spin. The A value for this is $A = 1.27 \times 10^{-4} \text{ sec}^{-1}$. If the spin-change does occur, then the resulting photon can ionize hydrogen, but since the timescale for this is long ($\tau \approx 2.2$ hours), it's possible that the atom will get collided out of 2^3S before it can decay. In fact, some of these collisions will dump the electron directly in 1^1S . If this occurs, an ionizing photon will be lost. Depending on the density, He I will eat up between 4% and 34% of hydrogen's ionizing photons.)

The accurate computation of ionization balance for a hydrogen-helium nebula can be moderately complicated, but the problem can be approximated by noting that $a_\nu(He^0)$ is always significantly larger than $a_\nu(H)$. Thus, if you make the assumption that all photons capable of ionizing helium do ionize helium, then

$$\int_{\nu_1}^{\infty} \frac{L_\nu}{h\nu} d\nu = Q(He^0) = \frac{4}{3}\pi N(He^+) N_e \alpha_B(He^0) \mathcal{R}_{He^+}^3 \quad (12.36)$$

where ν_1 is the ionization frequency of helium, and \mathcal{R}_{He^+} is the radius of the He I Strömngren sphere.

If the central star is hot enough to produce photons with energies greater than 54.4 eV (228 Å), then helium can become doubly ionized. Because of the extremely large differences between the energy levels of He II, a recombination to He II will produce, on average, 1.25 photons capable of ionizing hydrogen. So, while a small percentage of ionizing photons may be lost due to collision processes in He I, another small percentage can be recovered with He II.

Photo-ionization of Heavy Elements

An insignificant amount of the central star's photons go into ionizing the heavy elements. The photo-ionization cross section of these elements is complicated, since 1) the experimental values for the cross sections are not well determined, 2) the cross sections are not smooth functions, due to resonances, 3) other effects can place the electrons of metals into excited states, from which the ionization can occur. Metal recombination rates are similarly uncertain, due to effects such as dielectronic recombination. (Imagine a recombination where the photon produced via the recombination goes directly into exciting one of the ground-state electrons. If this happens, the atom can have two excited electrons at once, with a total excitation energy greater than the ionization energy. If this occurs, one electron may give its energy to the other, causing the atom to re-ionize itself!)

Below are some of the energies (in eV) needed to ionize metals:

Element	II	III	IV	V
Hydrogen	13.6			
Helium	24.6	54.4		
Carbon	11.3	24.4	47.9	64.5
Nitrogen	14.5	29.6	47.4	77.7
Oxygen	13.6	35.1	54.9	77.4
Neon	21.6	41.0	63.5	97.1

Note that to strip one electron off oxygen requires 13.6 eV — exactly the same as hydrogen. Thus, when hydrogen is ionized, oxygen is ionized.

Nebular Heating and Thermal Equilibrium

Consider the energy contained in the free electrons of a nebula. Every ionization creates a photo-electron with energy

$$\frac{1}{2}m_e v_p^2 = h(\nu - \nu_0) \quad (13.01)$$

while every recombination removes $\frac{1}{2}m_e v_r^2$. The difference between the mean energy of a photoionized electron, and that of a recombination electron must be lost by radiation if the nebula is to be in thermal equilibrium.

There is only one source of heating for a nebula: the photo-ionization of electrons. It is not difficult to estimate the initial temperature (T_i) of these electrons. For a pure hydrogen nebula ionized by a blackbody the average energy of a photoionized electron is

$$\frac{3}{2}kT_i = \frac{\int_{\nu_0}^{\infty} \frac{J_\nu}{h\nu} h(\nu - \nu_0) d\nu}{\int_{\nu_0}^{\infty} \frac{J_\nu}{h\nu} d\nu} \quad (13.02)$$

One can find an analytic expression for this, if one assumes that the ionizing spectrum is a blackbody, and all the ionizing photons are on the Wien part of the blackbody curve, *i.e.*, $kT_* < h\nu_0$. In this case,

$$J_\nu = B_\nu(T_*) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT} \quad (13.03)$$

If you make this substitution, and crank through the math, then you find that

$$\frac{3}{2}kT_i \approx kT_* \implies T_i \approx \frac{2}{3}T_* \quad (13.04)$$

Two things to note. First, T_i depends only the shape of the ionizing radiation, not on its strength. (In other words, the thermal energy of an ionized electron depends only on the energy of the photon that hit it, not on how many photons there are.) Thus, the temperature of a nebula being ionized by 100 large 30,000 K stars will be less than that ionized by one 100,000 white dwarf. (The size of the nebula will be larger, but the temperature will be smaller).

Second, because the cross section to photoionization decreases as ν^{-3} , high energy photons will penetrate deeper into a nebula than low energy photons. Thus, T_i may increase as the distance from the star increases.

The exact heating of a nebula is not difficult to compute. If $G(H)$ is the energy gained by the nebula through the photoionization of hydrogen, then

$$G(H) = N(H^0) \int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} h(\nu - \nu_0) a_\nu d\nu \quad (13.05)$$

which, through photoionization equilibrium, is

$$G(H) = N_e N_p \alpha_A \frac{\int_{\nu}^{\infty} \frac{J_\nu}{h\nu} h(\nu - \nu_0) a_\nu d\nu}{\int_{\nu}^{\infty} \frac{J_\nu}{h\nu} a_\nu d\nu} \quad (13.06)$$

Since the last term is simply injected energy,

$$G(H) = N_e N_p \alpha_A \left(\frac{3}{2} k T_i \right) \approx N_e N_p \alpha_A k T_* \quad (13.07)$$

A similar equation holds for helium, so the total heating is

$$G = G(H) + G(He^0) + G(He^1) \quad (13.08)$$

The amount of metals in the nebula is so little, that heating from those elements can easily be neglected.

Sources of Cooling

There are three different mechanisms a nebula has for cooling itself. The first is through thermal bremsstrahlung of the free electrons; the full expression for this is

$$L_{ff} = \frac{2^5 \pi e^6 Z^2}{3^{3/2} h m_e c^3} \left(\frac{2\pi k}{m_e} \right)^{1/2} T_e^{1/2} g_{ff} N_e N_+$$

$$= 1.42 \times 10^{-27} Z^2 T_e^{1/2} g_{ff} N_e N_+ \text{ ergs cm}^{-3} \text{ s}^{-1} \quad (13.09)$$

where g_{ff} is the gaunt factor. Since virtually all of these photons are longward of ν_0 (most are in the radio), all this free-free luminosity leaves the nebula. However, this energy is mostly inconsequential. Free-free emission is the least important of all energy loss mechanisms.

A second source of cooling comes from the recombination of electrons into hydrogen (and helium). This is computed simply from the kinetic-energy weighted recombination coefficient to each level of the atom

$$L_R(H) = N_e N_p kT \beta_A \quad (13.10)$$

where

$$\beta_A = \sum_{n=1}^{\infty} \sum_{L=0}^{n-1} \beta_{nL} \quad (13.11)$$

A typical value for β is $10^{-13} \text{ cm}^3 \text{ s}^{-1}$. Note that even though β represents the energy released by a recombined electron, it does not have the units of energy. By definition, the electron energy has been divided by kT , which is why (13.10) has a kT in the expression.

Recombination cooling is significant, but it does have a very interesting property. Recall from our discussion about recombination

(12.12) that, due to electrostatic focusing, slow moving electrons are more likely to recombine than fast moving electrons. Thus, re-combinational cooling will continually remove electrons from the low-energy tail of the Maxwellian distribution. Consequently, if this were the only source of nebular cooling, the temperature of the free electrons would actually increase!

This same effect makes the on-the-spot approximation for thermal balance less accurate than it is for ionization balance. (The slow moving electrons from the diffuse field are more important.) Still, one can use β_B , instead of β_A in (13.10) and get a reasonable approximation for re-combinational cooling.

Collisional Cooling

The most important physical process that occurs in an emission line region is collisional cooling. Consider an electron in some atom in state nL . If nL is not the ground state, that electron may decay, at a rate determined by the A value of the transition. If nL is the ground state, then, of course, the electron will just sit there. However, if the density of particles is high, or if the A value for downward transitions are low, a free electron may come by and collide the atom out of state nL into state $n'L'$ before a spontaneous transition can occur. If state $n'L'$ has less energy than state nL , then the free electron must carry away the energy. However, if state $n'L'$ has ΔE more energy than state nL (as it must, if the bound electron is already in its ground state), then the colliding electron will lose that amount of energy in the collision. The nebula will have therefore cooled.

The point to notice about this effect is where it occurs. The energy difference between the ground state of hydrogen and the first excited state is $\Delta E = 10.2$ eV. Therefore, for collisional cooling to be important for hydrogen, $kT \sim 10.2$ eV, or $T \sim 120,000$ K. Since this is much hotter than the temperature of the free electrons, collisions with neutral hydrogen atoms (which are, of course, rare) cannot cause an excitation and therefore cannot cool the nebula.

For helium, the energy difference between the ground state and the $n = 2$ state is even higher (19.8 eV), so likewise, ground-state helium collisions are unimportant. Collisional cooling on metals, however, is another story. Ions with electrons in partially-filled p-orbitals (such as oxygen, nitrogen, carbon, neon, argon, and sulfur) all have energy levels that are $\sim kT$ above their ground state. Thus, they are an extremely efficient cooling mechanism.

To understand how efficient these ions are, we must consider the rate at which these collisions occur. The computation of the rates has four terms:

- 1) a term which includes the density of electrons (N_e), and the density of ions (N_i).
- 2) a term which includes the volume being swept out by a free electron each second ($\sigma \propto v$) and the effects of electrostatic focusing ($\sigma \propto v^{-2}$). In terms of temperature, this means that the rate of collisions must go as

$$q \propto T_e^{-1/2} \quad (13.12)$$

- 3) a term which includes the Boltzmann factor. For a free electron with velocity v , if $\Delta E > m_e v^2/2$, then clearly, there is not enough energy in the collision to excite the electron. In terms of temperature, the larger the value of ΔE , the fewer the number of electrons that can cause an excitation. On the other hand, if the collision involves a downward transition, there is no energy barrier. Thus

$$q \propto e^{-\Delta E/kT} \quad \text{if } \Delta E > 0 \quad (13.13)$$

- 4) a term which involves the statistical weights. If level $n'L'$ is actually composed of 10 different states, then the collisional rate out of that level will be 10 times smaller than to a similar level with only one state. In other words, $q \propto 1/\omega_i$.
- 5) a (dimensionless) quantum mechanical cross section, which depends on the wave functions of the states, $q \propto \Omega(i, j)$.

If we put these terms together, the rate of collisions from state i to state j is

$$R_C = N_e N_i q_{ij} \quad (13.14)$$

where

$$\begin{aligned} q_{ij} &= \left(\frac{2\pi}{kT_e} \right)^{1/2} \frac{\hbar}{m_e^{3/2}} \frac{\Omega(i, j)}{\omega_i} \\ &= 8.629 \times 10^{-6} \frac{\Omega(i, j)}{\omega_i T_e^{1/2}} \text{ cm}^3 \text{ s}^{-1} \end{aligned} \quad (13.15)$$

if state j has lower energy than state i (*i.e.*, $\Delta E < 0$). If, on the other hand, $\Delta E > 0$, then

$$q_{ij} = 8.629 \times 10^{-6} \frac{\Omega(i, j)}{\omega_i T_e^{1/2}} e^{-\Delta E/kT} \text{ cm}^3 \text{ s}^{-1} \quad (13.16)$$

Interestingly, the quantum mechanical collision strength, Ω , is symmetrical, *i.e.*, $\Omega(i, j) = \Omega(j, i)$. This causes the two collisional rates to be related

$$q_{ij} = \frac{\omega_j}{\omega_i} e^{-\Delta E/kT} q_{ji} \quad (13.17)$$

Also, if either $S = 0$ or $L = 0$, then there is a simple relation for the collision strength between a term with a singlet level and a term containing several levels, *i.e.*,

$$\Omega(SLJ, S'L'J') = \frac{(2J' + 1)}{(2S' + 1)(2L' + 1)} \Omega(SL, S'L') \quad (13.18)$$

For example, let's look at the energy levels of doubly ionized oxygen. The ground state of O^{++} is a “p-squared” configuration, *i.e.*, the electrons are in $1s^2 2s^2 2p^2$. In the $2p^2$ shell are 5 different energy levels: the ground state term is at $L = 1$, and, depending on the spins of the electrons, the term has three different energy states, 3P_0 , 3P_1 , and 3P_2 . A couple of eV above these P states is a 1D_2 state, and about 3 eV above that is a singlet $L = 0$ state, 1S_0 . All these states have $n = 2$, so transitions between them are classically forbidden; in practice, their A values are $\lesssim 0.1 \text{ sec}^{-1}$.

Because the D state and the S state are both singlets, a single number can be given for the collision strength between 3P and 1D : in this case, $\Omega({}^3P, {}^1D) = 2.17$. The collision strength from the individual levels are therefore

$$\begin{aligned}\Omega({}^3P_2, {}^1D) &= \frac{5}{9}\Omega({}^3P, {}^1D) \\ \Omega({}^3P_1, {}^1D) &= \frac{3}{9}\Omega({}^3P, {}^1D) \\ \Omega({}^3P_0, {}^1D) &= \frac{1}{9}\Omega({}^3P, {}^1D)\end{aligned}\tag{13.19}$$

If the density of particles is low, then it is unlikely that a collisional de-excitation will occur before a spontaneous decay occurs. In this case, every collisional excitation creates a photon that escapes the nebula. The total collisional cooling for a given species is then

$$L_C = \sum_j N_e N_i q_{1,j} h\nu_{1,j}\tag{13.20}$$

where the sum is taken over all excited levels (typically 4), and $h\nu_{1,j}$ is the energy difference between the ion's ground state and state j .

Collisional Cooling and Detailed Balance

If every collisional excitation from the ground state to level j were followed (eventually) by the emission of a photon (or multiple photons) which leave the nebula, then the collisional cooling rate would be

$$L_C = N_e N_i q_{1j} \cdot h \nu_{1j} \quad (13.21)$$

But, in practice, collisional de-excitation also occurs. Consider a two-level atom of an ion. Let N_1 be the number of ions in the ground state, and N_2 be the number of ions in the excited state. In equilibrium, the number of electrons going out of N_1 is equal to the number of electrons entering level N_1 , either by collisions or by radiative decay. In other words,

$$N_e N_i q_{12} = N_e N_2 q_{21} + N_2 A_{21} \quad (13.22)$$

The relative level population is therefore

$$\left(\frac{N_1}{N_2} \right) = \frac{N_e q_{12}}{N_e q_{21} + A_{21}} \quad (13.23)$$

The collisional cooling rate for this two-level ion is therefore

$$L_C = N_2 A_{21} h \nu_{21} = \frac{N_1 N_e q_{12}}{N_e q_{21} + A_{21}} A_{21} h \nu_{21} \quad (13.24)$$

Let's look at this equation a bit more closely. If we divide both the numerator and denominator of (13.24) by A_{21} , then

$$L_C = N_e N_1 q_{12} h \nu_{12} \left\{ \frac{1}{\frac{N_e q_{21}}{A_{21}} + 1} \right\} \quad (13.25)$$

Now note the limits. As the electron density, $N_e \rightarrow 0$, collisional de-excitation becomes unimportant, and the cooling rate becomes

$$L_C \longrightarrow N_e N_1 q_{12} h \nu_{12} \quad (13.26)$$

In this case, the cooling is proportional to the electron density, and every collision upward creates a photon which cools the nebula. On the other hand, as $N_e \rightarrow \infty$,

$$L_C \longrightarrow N_1 \left(\frac{q_{12}}{q_{21}} \right) A_{21} h \nu_{21} = N_1 \left(\frac{\omega_2}{\omega_1} \right) e^{-\Delta E/kT} A_{21} h \nu_{21} \quad (13.27)$$

This is nothing more than the cooling rate for a gas in thermodynamic equilibrium.

Of course, although a few atoms (such as Be II) can be treated with two levels, most require more. In fact, the p^2 , p^3 , and p^4 configurations all have 5 low-lying levels, separated with two major divisions. To calculate the cooling from an n -level atom, you have to solve a set of linear equations, equating the number of electrons entering a level to the number of electrons exiting the level.

$$\sum_{j \neq i} N_e N_j q_{ji} + \sum_{j > i} N_j A_{ji} = \sum_{j \neq i} N_e N_i q_{ij} + \sum_{j < i} N_i A_{ij} \quad (13.28)$$

The first sum is the number of electrons entering level i through collisions from all the other levels. The second term is the number of electrons entering level i through radiative decays from the levels above i . On the other side of the equal sign is the number of electrons exiting level i via collisions, and the number of electrons exiting level i via radiative decays downward. This is the equation of **detailed balance** for a collisionally excited line.

Note that there is actually one other equation to detailed balance

$$\sum_i^n N_i = N \quad (13.29)$$

This simply says that the sum of the ions in all the levels must add up to the total number of ions there are. So this gives you $n + 1$ equations, and n unknowns. Thus, with some algebra (and a little bit of patience) you can solve for the number of ions in each level.

Once the ion densities are known, the collisional cooling follows simply from

$$L_C = \sum_i N_i \sum_{j < i} A_{ij} h\nu_{ij} \quad (13.30)$$

The equation of detailed balance also allows us to specify when collisions out of a level are more important than radiative transfers out of a level. From the left side of (13.28), this occurs when

$$\sum_{j \neq i} N_e N_j q_{ji} > \sum_{j < i} N_j A_{ji} \quad (13.31)$$

or, more to the point,

$$N_e(\text{crit}) > \frac{\sum_{j < i} A_{ij}}{\sum_{j \neq i} q_{ij}} \quad (13.32)$$

$N_e(\text{crit})$ is called the critical density for the level. If $N_e < N_e(\text{crit})$, then collisional de-excitation is important. For most forbidden lines, this occurs, very roughly, at $N_e \sim 10^6 \text{ cm}^{-3}$.

Thermal Equilibrium

A nebula in thermal equilibrium has

$$G = L_{ff} + L_R + L_C \quad (13.33)$$

where G is the heating due to photoionization (equations 13.06 and 13.08), L_{ff} is the cooling from free-free emission (13.09), L_R is the cooling from electron recombination (13.10), and L_C is the cooling due to collisional excitations (13.20). In the low density limit, each of these terms is proportional to electron density, so N_e cancels out. In this case, the nebular temperature will only depend on the energy of the ionizing photons and on the ionic abundances. At high densities, however, collisional de-excitations will decrease the efficiency of collisional cooling. Therefore, all things being equal, denser nebulae will be hotter.

One very important point to consider is that collisions are, by far, the most important mechanism for nebular cooling. Naturally, the amount of collisional cooling is proportional to the metal abundance, N_Z . But note: if N_Z is decreased, then the temperature of the nebula will rise considerably. Now recall that the equation for collisional excitation is

$$q_{ij} = 8.629 \times 10^{-6} \frac{\Omega(i, j)}{\omega_i T_e^{1/2}} e^{-\Delta E/kT} \text{ cm}^3 \text{ s}^{-1} \quad (13.16)$$

so if T rises, the exponential term $e^{-\Delta E/kT}$ will also rise, and therefore q_{ij} will rise. As a result, it is possible that decreasing the number of ions will actually increase the amount of ionic emission from the nebula.

Collisional cooling is always most efficient for ions where $\Delta E \approx kT$. If $\Delta E \gg kT$, then very few electrons will be excited, so little cooling will occur; if $\Delta E \ll kT$, then the amount of energy released in the downward transition will be inconsequential.

The Emitted Spectrum of a Strömgren Sphere

There are four types of emission from a Stromgren sphere

- 1) Strong recombination lines of hydrogen and helium.
- 2) Very strong collisionally excited lines of metals. In the optical, these are mostly forbidden lines of oxygen, nitrogen, sulfur, argon, and neon. Lines of carbon, magnesium, and silicon appear in the UV.
- 3) There is some weak continuum emission associated with nebulae, due to free-free emission, free-bound emission, and two-photon emission. (The latter occurs when a $2s$ electron of hydrogen or helium decays to the $1s$ state by temporarily making an intermediate state for itself.) At most wavelengths, this is much, much fainter than the line emission (although, at radio wavelengths, the free-free and free-bound emission is significant).
- 4) Other weak emission lines due to resonance-fluorescence (where a recombination line of hydrogen happens to have the same wavelength of as a transition to the ground state of a metal), recombination of metals, and other odd effects.

Note that the size of a Strömgren Sphere can be defined in two ways. In a **radiation bounded** nebula, all the ionizing photons are absorbed by hydrogen. Alternatively one can have a **matter bounded** nebula, in which some ionizing photons escape into space simply because there are no hydrogen atoms around for them to ionize.

The Hydrogen Spectrum

Since the A values for hydrogen are $A \sim 10^6 \text{ sec}^{-1}$, while the q values are 10^{-4} , collisional excitations for hydrogen only become important when

$$N_e \gtrsim \frac{A}{q} \sim 10^{10} \text{ cm}^{-3} \quad (14.01)$$

Thus, with slight exceptions for electrons in extremely high levels ($n \sim 100$), collisions do not affect the hydrogen atom. One can therefore write the detailed balance equation for the level populations of hydrogen as

$$N_p N_e \alpha_{nL}(T) + \sum_{n' > n}^{\infty} \sum_L N_{n'L'} A_{n'L',nL} = N_{nL} \sum_{n'=1}^{n-1} \sum_L A_{nL,n'L'} \quad (14.02)$$

where $\alpha_{nL}(T)$ is the recombination coefficient representing the rate at which recombinations occur directly into state nL . In other words, the number of recombinations directly into a level plus the number of radiative decays into that level is equal to the number of radiative decays out of the level. If (14.02) is correct, then the emission produced by any transition is

$$j_{nn'} = \frac{h\nu_{nn'}}{4\pi} \sum_L \sum_{L'=L\pm 1}^{n-1} N_{nL} A_{nL,n'L'} \quad (14.03)$$

where we have explicitly kept the individual L states separate, although they are degenerate in energy (and therefore summed for the total emission).

Equation (14.02) represents a set of $n \times L$ linear equations with $n \times L$ unknowns. In theory, one can choose an electron temperature and density, look up the values of α_{nL} appropriate for the

temperature, and then solve for the population levels with linear algebra. Once the level populations are known, the emission strength of each hydrogen line follow through (14.03).

In practice, the level populations are only weakly dependent on N_e and T_e (*i.e.*, direct recombination is only one of N -ways into $n = 2$). Consequently, the level populations, and therefore the line ratios are almost entirely determined by the Einstein A values, not N_e or T_e .

Equation (14.02) explicitly neglects two effects. The first is collisional excitations and de-excitations. For the ground state of hydrogen, $\Delta E \ll kT_e$, so free electrons don't have enough energy to cause a collisional excitation. And, while electrons in excited states can be affected by collisions, the critical density for this is $N_e \gtrsim 10^9 \text{ cm}^{-3}$. Hence, in most cases, collisions can be neglected. (Actually, this is not strictly true: because it doesn't take much of a collision to move electrons from say, $n=101$ to $n=100$, in the electrons at these high levels can, and will, be redistributed. The result will be more like a Boltzmann distribution for these levels.)

The other piece of physics missing from (14.02) is input to levels via absorptions from lower levels. Since hydrogen electrons decay to the ground state in $\sim 10^{-6} \text{ sec}^{-1}$, virtually no absorptions occur from these states. However, absorptions from the ground state can and do occur.

To understand the effects of absorption, let's first consider the cross section of the hydrogen atom to Lyman photons.

Line	Wavelength (Å)	A (sec ⁻¹)	a_0 (cm ²)	$\tau_0/\tau_{912 \text{ Å}}$
Ly α	1215.67	6.26×10^8	5.90×10^{-14}	9366
Ly β	1025.72	1.67×10^8	9.46×10^{-15}	1501
Ly γ	972.54	6.82×10^7	3.29×10^{-15}	522
Ly 10	920.96	4.21×10^6	1.72×10^{-16}	27
Ly 15	915.82	1.24×10^6	5.00×10^{-17}	8
Ly 20	914.04	5.24×10^5	2.10×10^{-17}	3

Especially for the lower Lyman lines, the cross section for absorption is *much greater* than it is for an ionization. (Recall that at 912 Å, the ionization cross section is 6.30×10^{-18} cm².) So, if the on-the-spot approximation is valid, Ly absorptions must also be important.

Traditionally, two types of approximations are used. Nebulae are divided into two types:

- 1) CASE A: all Lyman line photons produced by recombining hydrogen atoms escape the nebula. In this case, no Lyman absorptions occur, and equation (14.02) is strictly valid. This is the optically thin case.
- 2) CASE B: all Lyman line photons are re-absorbed by other hydrogen atoms. In this case, all downward transitions to the ground state don't count (in the same way as ground state recombinations don't count in the on-the-spot approximation). In this optically thick case, equation (14.02) is still valid, except transitions to the $n=1$ level are not included in the summations (since, in effect, they never occur).

In real life, almost everything is CASE B (or close to it), at least for the lower Lyman lines.

CASE B has an interesting implication for astrophysics. According to ionization balance, every ionization results in a recombination (with recombinations to the ground level not counted). Once in the hydrogen atom, the electrons decays downward, but decays to the ground state don't count. As a result, every decay must eventually go to the $n = 2$ state (since it can't go to the $n = 1$ state). All these transitions are in the optical! Thus, the net result is every ionization produces an optical Balmer line photon. By counting the number of Balmer photons, you can count the number of ionizing photons emitted from the central star.

The equations are even more elegant. The level populations given by (14.02) depend almost exclusively on the A values of the transitions, and are only weakly dependent to N_e and T_e . Since the level populations are (more-or-less) fixed by atomic physics, the line ratios are also (more-or-less) fixed by atomic physics. Thus, if you measure one Balmer line, you more-or-less know all the Balmer line strengths. So, by measuring one Balmer line (say, $H\beta$), you can derive the number of ionizing photons coming from the central star.

Below is a table giving the line ratios for the lowest level Balmer transitions.

Temperature N_e (cm^{-3})	5000 10^4	10,000 10^2 10^6		20,000 10^2 10^4	
$\alpha_{\text{H}\beta}^{\text{eff}}$	5.44	3.02	3.07	1.61	1.61
$I(\text{H}\alpha)/I(\text{H}\beta)$	3.00	2.86	2.81	2.75	2.74
$I(\text{H}\gamma)/I(\text{H}\beta)$	0.460	0.468	0.471	0.475	0.476
$I(\text{H}\epsilon)/I(\text{H}\beta)$	0.155	0.159	0.163	0.163	0.163

$\alpha_{\text{H}\beta}^{\text{eff}}$ in units of $\text{cm}^3 \text{ s}^{-1}$.

In addition to line ratios, the table also has a value called $\alpha_{\text{H}\beta}^{\text{eff}}$. Since the recombination coefficient for hydrogen is known, and since all the line ratios for hydrogen are known, it is a relatively straightforward task to combine these numbers and figure out how many recombinations eventually create an $\text{H}\beta$ photon. This is called the “effective” recombination coefficient for $\text{H}\beta$. In general,

$$N_e N_p \alpha_{nn'}^{\text{eff}} = \sum_{L=0}^{n-1} \sum_{L'=L\pm 1} N_{nL} A_{nL,n'L'} = \frac{4\pi j_{nn'}}{h\nu_{nn'}} \quad (14.04)$$

Continuum Emission

There are three sources of continuum emission from a nebula

- 1) Free-free emission. Most of this light emerges in the radio, although some optical continuum is produced.
- 2) Free-bound transitions (*i.e.*, recombinations). Recall that most recombinations involve slowly moving electrons, *i.e.*, electrons with energies just about 13.6 eV. Recall also that there are many more high n levels for hydrogen than low n levels. As a result, most of this radiation comes out at long wavelengths, though just to the blue of the ionization edge of each level, there is a large jump in the amount of continuum flux emitted.
- 3) 2-photon emission. This mechanism comes from relativistic quantum mechanics. Recall that the Hamiltonian operator for an electron is

$$\mathbf{H} = \frac{\vec{p}^2}{2m_e} + \phi(x, t) \quad (14.05)$$

where \vec{p} is the electron momentum and ϕ is the scalar potential in the atom. Relativistically, however, the momentum is given by

$$\vec{P} = \vec{p} - \frac{q}{c}\vec{\Phi}(x, t) \quad (14.06)$$

where Φ is the vector potential of the radiation field, and $q = -e$ the electron charge. Thus, the Hamiltonian operator becomes

$$\mathbf{H} = \left| \vec{p} + \frac{e}{c}\vec{\Phi} \right|^2 - e\phi \quad (14.07)$$

or

$$\mathbf{H} = \left(\frac{1}{2m_e} |\vec{p}|^2 - e\phi \right) + \frac{e}{2m_e c} \left(\vec{p} \cdot \vec{\Phi} + \vec{\Phi} \cdot \vec{p} \right) + \frac{e^2}{2m_e c^2} \left(\vec{\Phi} \cdot \vec{\Phi} \right) \quad (14.08)$$

The first term describes the atom in equilibrium, and the second term governs interactions between the atom and a radiation field. The third, quadratic, term describes interactions between the atom and two photons at once. It is much smaller than the second term (by about 10 orders of magnitude), but this is the only process by which an electron in the $2s$ state of hydrogen can radiatively decay down to the $1s$ state.

If the difference between the $n=1$ and $n=2$ states of hydrogen is 10.2 eV, then, by definition, the two photon emission process must create must as many photons shortward of 5.1 eV (2431 Å) as longward. Most of this emission is therefore in the ultraviolet. Note, however, that since the A value for this transition is 8.23 sec^{-1} , in higher-density nebulae, electrons can be collided out of the $2s$ state before they can radiatively decay via this process.

Ly α and Resonance Fluorescence

Under ionization balance, every ionization is balanced by a recombination, and under Case B, every hydrogen recombination eventually results in a Balmer photon (which leaves the nebula). These $n=2$ electrons will, of course, decay to $n=1$, but the Ly α photons that are produced are trapped — they just get re-absorbed (and re-emitted) by other neutral atoms. This can happen hundreds or even thousands of times, but eventually, something has to happen to all these Ly α photons.

Actually, there are several possible ways to get rid of Ly α . The principle ones are

- 1) Ly α may “leak” out of the nebula. These photons can random walk their way to the edge of the nebula, then escape, or have their wavelength shifted slightly due to the quantum mechanics of the hydrogen atom (*i.e.*, they can be emitted “in the wings” of the line). The process is assist if the nebula has bulk motions, which can redshift a Ly α photon out of the line-center.
- 2) Ly α may get destroyed by hitting a dust grain. There’s usually some dust floating around, and if a photon hits it, the photon will be destroyed (and the grain will be heated). One dust grain can kill many, many Ly α photons.
- 3) The electron may be collided out of the $n=2$ P-state into either the $n=1$ state, or the $n=2$ S-state. In the latter case, the electron may (if it doesn’t get hit again) decay via 2-photon emission.
- 4) A “resonance fluorescence” occurs. Actually, this doesn’t happen with Ly α per se, but with Ly β (destroying the photon before it gets to $n=2$) or with the Ly α of He II. (Helium has the same problem as hydrogen, and has its own Case B set of equations.)

The way resonance fluorescence works is this. Let's take He II Ly α as an example. The wavelength of this Ly α transition is 303.78 Å. A permitted, ground-state transition of O⁺⁺ ($3d\ ^3P_2 \longrightarrow 2p^2\ ^3P_2$) is 303.80 Å. This coincidence can cause the helium Ly α photon to excite oxygen instead of a helium ion. The oxygen atom can then decay (via another route) and emit photons of various wavelengths that escape the nebula. (There are a whole bunch of permitted oxygen emission lines between 2800 Å and 3800 Å that are produced in this way.)

Resonance fluorescence is more interesting than it is useful. Unless one is doing detailed radiative transfer calculations, it can usually be ignored.

Nebular Diagnostics

Ions with ground state configurations of p^2 , p^3 , and p^4 have four low-lying levels with energy separations of the order of kT . The p^2 and p^4 configurations are especially good for determining electron temperatures; ions with this configuration include N^+ , O^{++} , Ne^{+4} , S^{++} , and Ar^{+4} (all p^2) and O , Ne^{++} , and Ar^{++} (p^4). Ions with p^3 configurations, such as O^+ , Ne^{+++} , S^+ , and Ar^{+++} , are useful for measuring density.

The flux generated by any forbidden transition (from i to j) is

$$F_{ij} = N_i A_{ij} h\nu_{ij} \quad (14.09)$$

where N_i is the number of ions in the i state, A_{ij} is the Einstein A value of the transition, and $h\nu_{ij}$ the energy difference of the transition. Also recall that the number of ions in state i is given by the conditions of detailed balance

$$\sum_{j \neq i} N_e N_j q_{ji} + \sum_{j > i} N_j A_{ji} = \sum_{j \neq i} N_e N_i q_{ij} + \sum_{j < i} N_i A_{ij} \quad (13.28)$$

In other words, the number of collisions into the level plus the number of radiative decays into the level is equal to the number of collisions out of the level plus the number of decays out of the level. The collisional rates, q_{ij} are given by

$$q_{ij} = 8.629 \times 10^{-6} \frac{\Omega(i, j)}{\omega_i T_e^{1/2}} \text{ cm}^3 \text{ s}^{-1} \quad \text{for } i > j \quad (13.15)$$

and

$$q_{ij} = 8.629 \times 10^{-6} \frac{\Omega(i, j)}{\omega_i T_e^{1/2}} e^{-\Delta E/kT} \text{ cm}^3 \text{ s}^{-1} \quad \text{for } i < j \quad (13.16)$$

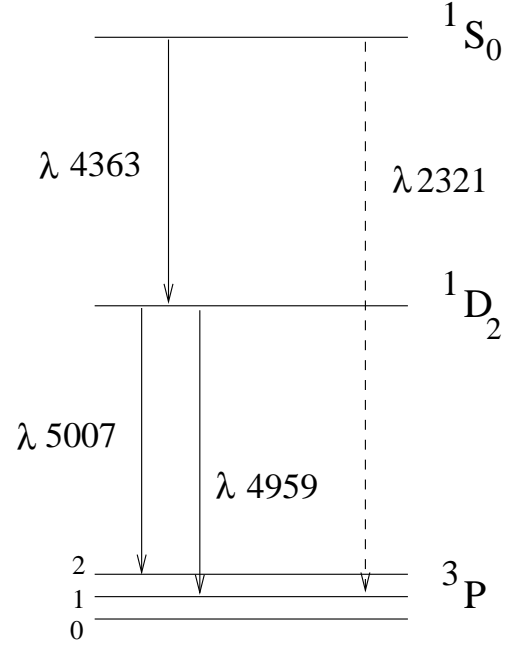
Thus, for any T_e and N_e , one can solve for N_i , and thus for the F_{ij} . By taking the ratios of certain lines, measurements of T_e and N_e can be made without knowing distance or abundance.

Estimating the Electron Temperature

Shells with p^2 or p^4 configurations have a 3P term for the ground state, a singlet D term $\Delta E \sim kT$ above the ground state, and a singlet S term with $\Delta E \sim kT$ above the D state. This type of 3-1-1 splitting works well for temperature determinations.

Consider the O^{++} ion. The $\lambda 5007$ line and the $\lambda 4959$ both start at the 1D_2 level, so the ratio of these two lines is simply the ratio of their A values. Since the A value for $\lambda 5007$ is 3 times that of the $\lambda 4959$ transition, $I(5007)/I(4959) = 3$. Similarly, the fraction of 1S_0 ions that produce optical ($\lambda 4363$) light rather than UV ($\lambda 2321$) light is

$$f = \frac{A_{4363}}{A_{4363} + A_{2321}} \quad (14.10)$$



(Actually, the denominator of equation (15.10) should also include the A values to the 3P_2 and 3P_0 states, but these rates are negligible compared to the other two.)

Now the ratio of $^3P \rightarrow ^1D$ collisions to $^3P \rightarrow ^1S$ collisions is

$$\begin{aligned} R &= \frac{N_e N_{^3P} \left(\frac{8.629 \times 10^{-6}}{T^{1/2}} \right) \frac{\Omega(^3P, ^1D)}{\omega_{^3P}} e^{-\Delta E_{^3P, ^1D}/kT}}{N_e N_{^3P} \left(\frac{8.629 \times 10^{-6}}{T^{1/2}} \right) \frac{\Omega(^3P, ^1S)}{\omega_{^3P}} e^{-\Delta E_{^3P, ^1S}/kT}} \\ &= \frac{\Omega(^3P, ^1D)}{\Omega(^3P, ^1S)} e^{\Delta E_{^1D, ^1S}/kT} \end{aligned} \quad (14.11)$$

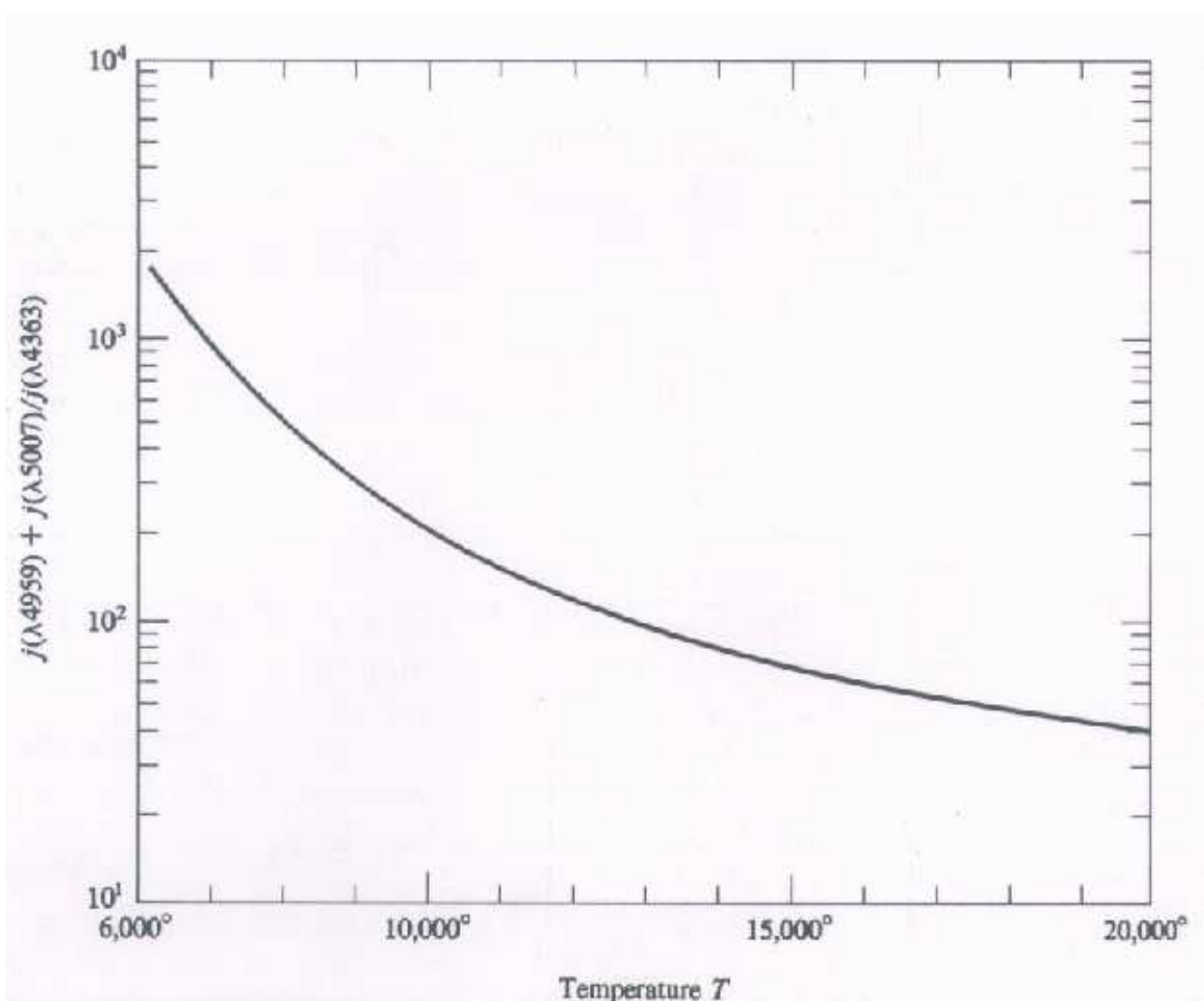
If the number of radiative decays into 1D_2 from 1S_0 is small compared to the number of collisions into 1D_2 from the ground state, then the intensity of each line is simply proportional to the number of collisions into the level. So if we combine (14.10) with (14.11), then

$$\frac{I(\lambda 4959) + I(\lambda 5007)}{I(\lambda 4363)} = \frac{\Omega(^3P, ^1D)}{\Omega(^3P, ^1S)} \frac{A_{4959} + A_{5007}}{A_{4363}} \frac{\bar{\nu}}{\nu_{4363}} e^{\Delta E/kT} \quad (14.12)$$

where ΔE is the energy difference between the 1D_2 state and the 1S_0 state, and $\bar{\nu}$ is the “mean” frequency of a photon produced by a decay from 1D_2 , *i.e.*,

$$\bar{\nu} = \frac{\nu_{4959}A_{4959} + \nu_{5007}A_{5007}}{A_{4959} + A_{5007}} \quad (14.13)$$

This expression, of course, assumes that all the D and S electrons come from collisions with ground state electrons. It therefore is strictly true only in the low-density limit. At higher densities, ($N_e \gtrsim 10^5$), the line ratios must be solved via the detailed balance equations.

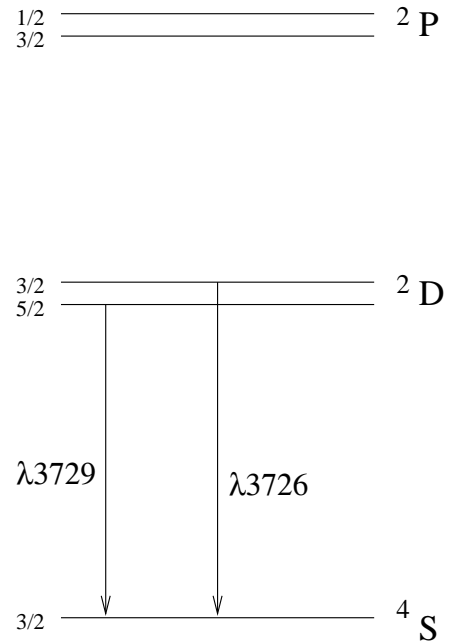


The [O III] lines intensity ratio (in the low density limit) as a function of temperature. Note that at low temperatures, very few electrons are collisionally excited up to S-term level. Consequently, the $\lambda 4363$ line is extremely weak. Similar relations exist for other ions with p^2 or p^4 configurations.

Estimating the Electron Density

Ions with p^3 configurations have a pair of closely spaced 2D levels $\Delta E \sim kT$ above the ground state, and another pair of 2P levels $\Delta E \sim kT$ above the D term. We can use the 1-2-2 splitting to measure the electron density.

To see how, consider the O^+ ion. The 2D term is split into two levels with essentially the same energy above the ground state. The ${}^2D_{3/2}$ level has a statistical weight of $\omega = (2 \times \frac{3}{2}) + 1 = 4$; the ${}^2D_{5/2}$ level has $\omega = 6$. Since the ground state of a p^3 configuration is an S state (*i.e.*, $L = 0$), the collision strength from the ground state is the same for both levels, except for the statistical weight (see equation 13.18).



So, if Ω is the total collision strength from S to D , the collision strength from S to ${}^2D_{5/2}$ is 0.6Ω , and that from S to ${}^2D_{3/2}$ is 0.4Ω . (In other words, according to the definition of statistical weights, six-tenths of the D states have $J = 5/2$, and four-tenths have $J = 3/2$: the electrons don't care which state they enter, and there are more $J = 5/2$ states than $J = 3/2$ states.)

Now, in the low density limit, every collision upward results in a radiative decay downward. The ratio of ${}^2D_{5/2} \rightarrow {}^4S_{3/2}$ decays to ${}^2D_{3/2} \rightarrow {}^4S_{3/2}$ is then simply the ratio of the number of collisions into each level. Since the energies of both states are essentially identical, the ratio of the lines is just the ratios of Ω , which are, in turn, just the ratios of the statistical weights. Therefore, in the

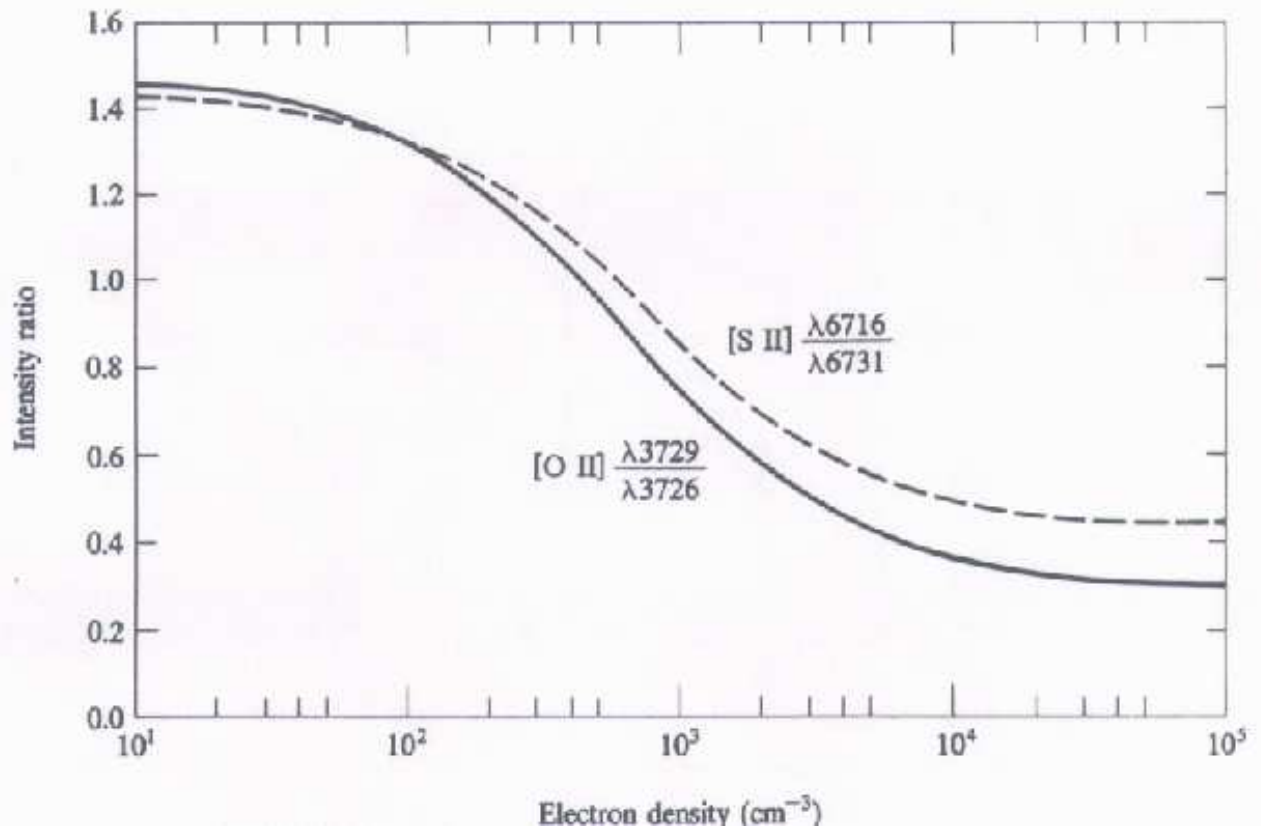
low density limit,

$$\frac{I(\lambda 3729)}{I(\lambda 3726)} = \frac{\omega_{3729}}{\omega_{3726}} = \frac{6}{4} = 1.5 \quad (14.14)$$

In the high density limit, however, things are different: electron collisions are continually populating and depopulating the D states. Again, since the statistical weight of ${}^2D_{5/2}$ is 1.5 times that of ${}^2D_{3/2}$, all things being equal, there will be 1.5 more decays from that level. However, since each level has an infinite supply of electrons from collisions, the ratio of the lines also depends on the ratio of the A values (see equations 14.09 and 14.10). (Imagine if one level had an A value ten times that of the other; in that case, 10 photons would be produced in the time it takes the other level to produce 1 photon.) Thus, in the high density limit

$$\frac{I(\lambda 3729)}{I(\lambda 3726)} = \frac{\omega_{3729} A_{3729}}{\omega_{3726} A_{3726}} = 1.5 \frac{A_{3729}}{A_{3726}} \quad (14.15)$$

In the intermediate case, of course, one must solve for the line ratios via the equations of detailed balance. However, the ratio of the two lines is smoothly function, so simple interpolations will work.



The variation of the intensity ratio of [O II] $\lambda 3729$ to $\lambda 3726$, and of [S II] $\lambda 6716$ to $\lambda 6731$ as a function of density at $T = 10,000$ K. At other temperatures, the curves are very nearly identical, if the x-axis is defined as $N_e / \sqrt{T/10,000}$.

Measuring the Ionizing Flux of Photons

In the optical thick CASE B is valid, then it is easy to use the recombination lines to measure the ionizing flux from the central star (or stars). Recall that from ionization equilibrium

$$\int_{\nu_0}^{\infty} \frac{L_{\nu}}{h\nu} d\nu = Q(\text{H}^0) = \int_0^{\mathcal{R}} N_e N_p \alpha_B dV \quad (12.35)$$

where V is the nebular volume. (In the spherical nebula case, $dV = 4\pi r^2 dr$.) Also recall that under CASE B, every ionization causes a recombination which creates a Balmer photon that escapes the nebula. The $\text{H}\beta$ luminosity is therefore

$$L(\text{H}\beta) = \int_0^{\mathcal{R}} 4\pi j_{\text{H}\beta} dV = h\nu_{\text{H}\beta} \int_0^{\mathcal{R}} N_e N_p \alpha_{\text{H}\beta}^{eff} dV \quad (14.16)$$

The ratio of ionizing photons to $\text{H}\beta$ photons is therefore

$$\frac{L(\text{H}\beta)/h\nu_{\text{H}\beta}}{\int_{\nu_0}^{\infty} \frac{L_{\nu}}{h\nu} d\nu} = \frac{\int_0^{\mathcal{R}} N_e N_p \alpha_{\text{H}\beta}^{eff} dV}{\int_0^{\mathcal{R}} N_e N_p \alpha_B dV} \quad (14.17)$$

In other words, the ratio of $\text{H}\beta$ photons to ionizing photons is the density weighted average of $\alpha_{\text{H}\beta}^{eff}$ divided by the density weighted average of α_B . So

$$\frac{L(\text{H}\beta)/h\nu_{\text{H}\beta}}{\int_{\nu_0}^{\infty} \frac{L_{\nu}}{h\nu} d\nu} \approx \frac{\alpha_{\text{H}\beta}^{eff}}{\alpha_B} \quad (14.18)$$

Both $\alpha_{\text{H}\beta}^{eff}$ and α_B are physical functions that depend only on temperature (in a similar fashion). Thus, given an estimate for the nebular temperature, the constant relating $\text{H}\beta$ photons to ionizing photons is easily found.

Measuring the Temperature of an Ionizing Source

If CASE B holds, and the ionizing source is a single star, we can use the nebular $H\beta$ flux and the V -band magnitude of the central star to measure the central star's temperature. First, let's approximate the flux distribution of the ionizing star as a blackbody, $B_\nu(T)$. The ratio of V -band photons to ionizing photons will clearly be a function of temperature. (You can compute this on a calculator, if you like.) Let's call that function $F(T)$. From (14.18)

$$F(T) = \frac{B_\nu(T)}{\int_{\nu_0}^{\infty} \frac{B_\nu(T)}{h\nu} d\nu} = \frac{\mathcal{L}_V}{\int_{\nu_0}^{\infty} \frac{\mathcal{L}_\nu}{h\nu} d\nu} = \frac{\mathcal{L}_V}{\frac{L(H\beta)}{h\nu_{H\beta}} \frac{\alpha_{H\beta}^{eff}}{\alpha_B}} \quad (14.19)$$

or

$$F(T) = \frac{\mathcal{L}_V}{\mathcal{L}(H\beta)} \frac{\alpha_B}{\alpha_{H\beta}^{eff}} h\nu_{H\beta} \quad (14.20)$$

We can easily convert total luminosity to observed flux by dividing both numerator and denominator by distance squared. Therefore

$$F(T) = \frac{f_V}{f(H\beta)} \frac{\alpha_B}{\alpha_{H\beta}^{eff}} h\nu_{H\beta} \quad (14.21)$$

In theory, the observed V -band flux of the central star and the observed $H\beta$ flux from the nebula are both visible. One can therefore find the value of the right-hand side of the equation, and look up the temperature of the source via the function $F(T)$. This is called the Zanstra method for measuring stellar temperatures.

Variants of the Zanstra method work for He I and He II as well. (He I has a line at $\lambda 5876$ which serves the same function as $H\beta$; the He II equivalent of $H\beta$ is a line at $\lambda 4686$.)

Estimating Extinction to a Nebula

Estimating the extinction to a nebula is almost trivial. The ratio of the Balmer emission lines is extremely insensitive to density and temperature. You cannot go too far wrong if you assume $H\alpha/H\beta \approx 2.86$. If the ratio is larger than this, the discrepancy is almost certainly due to differential extinction, *i.e.*, reddening. Recall that the amount of extinction (in magnitudes or log flux) at any (optical) wavelength is $A_\lambda \propto 1/\lambda$. According to this $1/\lambda$ extinction law

$$\log f_\lambda^{obs} = \log f_\lambda^0 - \frac{a}{\lambda} \quad (14.22)$$

where f_λ^{obs} represents the observed flux at any wavelength, f_λ^0 is the de-reddened flux, and a is a constant that represents the amount of extinction. Now, for consistency with the literature, let c equal the total logarithmic extinction at $H\beta$, *i.e.*, $c = a/\lambda_{4861}$. So

$$\log f_{H\beta}^{obs} = \log f_{H\beta}^0 - c \quad (14.23)$$

In terms of c , the ratio of the observed flux at any wavelength λ to the observed flux of $H\beta$ is then

$$\log \left(\frac{f_\lambda^{obs}}{f_{H\beta}^{obs}} \right) = \log \left(\frac{f_\lambda^0}{f_{H\beta}^0} \right) - c \left(\frac{\lambda}{\lambda_{H\beta}} - 1 \right) \quad (14.24)$$

Since the intrinsic $H\alpha/H\beta$ ratio is known, (14.24) can immediately be solved for c . Once c is known, the de-reddened $H\beta$ flux follows immediately from (14.23), and, with c and $f_{H\beta}^0$ both known, (14.24) can be used to solve for the de-reddened flux at any wavelength.

Deriving Nebular Abundances

The key to deriving nebular abundances is to use the nebular diagnostics to fix the electron temperatures and densities. Once that is done, the abundance of any species follows simply from the observed fluxes. For instance, for hydrogen, He I, and He II, the amount of emission is

$$I(\text{H}\beta) = N_e N_p \alpha_{\text{H}\beta}^{eff} \quad (14.25)$$

$$I(\lambda 5876) = N_e N_{\text{He}^+} \alpha_{\lambda 5876}^{eff} \quad (14.26)$$

$$I(\lambda 4686) = N_e N_{\text{He}^{++}} \alpha_{\lambda 4686}^{eff} \quad (14.27)$$

while for collisionally excited ions, the amount of emission is

$$I_{ij} = N_i N_e q_{ij} \quad (14.28)$$

Since the values of α and q_{ij} are functions of temperature and density only, once those values are known, the line strengths become directly proportional to abundance.

Note that if one wants to measure the total abundance of oxygen, one needs to add up the abundances of oxygen in all its states, *i.e.*, O, O⁺, O⁺⁺, *etc.* Fortunately, by observing the strengths of the recombinational lines of hydrogen, He I, and He II one can get a pretty good idea of the underlying distribution of ionizing photons. Thus, one can estimate the relative abundance of, say, O⁺³ to O⁺², even if no lines are O⁺³ are observable.

One problem that arises in nebular analysis is that there is quite a lot of redundancy in the information provided by the emission lines. In some cases, not all the measurements of density and temperature are consistent. In particular, the density measurements from the [O II] lines are sometimes much larger than those measured via other techniques (specifically, radio observations of the nebular continuum). This has led to the introduction of f , the filling factor. Many nebulae are therefore modeled by assuming the existence of clumps of material with density N_e separated by regions with density zero. This value f finds its way into many of the equations presented, *i.e.*, the equation of ionization balance becomes

$$Q(\text{H}^0) = \frac{4}{3}\pi\mathcal{R}^3 N_e N_p \alpha_B \cdot f \quad (14.29)$$